

# Notes on “quantum gravity” and noncommutative geometry

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**Abstract** I hesitated for a long time before giving shape to these notes, originally intended for preliminary reading by the attendees to the Summer School “New paths towards quantum gravity” (Holbaek Bay, Denmark, May 2008). At the end, I decide against just selling my mathematical wares, and for a survey, necessarily very selective, but taking a global phenomenological approach to its subject matter. After all, noncommutative geometry does not purport yet to solve the riddle of quantum gravity; it is more of an insurance policy against the probable failure of the other approaches. The plan is as follows: the introduction invites students to the fruitful doubts and conundrums besetting the application of even classical gravity. Next, the first experiments detecting quantum gravitational states inoculate us a healthy dose of scepticism on some of the current ideologies. In Section 3 we look at the action for general relativity as a consequence of gauge theory for quantum tensor fields. Section 4 briefly deals with the unimodular variants. Section 5 arrives at noncommutative geometry. I am convinced that, if this is to play a role in quantum gravity, commutative and noncommutative manifolds must be treated on the same footing; which justifies the place granted to the reconstruction theorem. Together with Section 3, this part constitutes the main body of the notes. Only very summarily at the end of this section we point to some approaches to gravity within the noncommutative realm. The last section delivers a last dose of scepticism. My efforts will have been rewarded if someone from the young generation learns to mistrust current mindsets.

## 1 Introduction

“Quantum gravity” denotes a problem, not a theory. There is no theory of quantum gravity. There exist several competing schemes, as mathematically sophisticated and fecund, as a rule, as undeveloped in the face of experimental evidence and of the purported aim of unifying gravity with other fundamental interactions.

My account of the subject is unabashedly *low-road*. The concept was coined by Glashow in his thought-provoking book [1]. The low road:

... is the path from the laboratory to the blackboard, from experiment to theory, from hard-won empirical observations to the mathematical framework in which they are described, explained and ultimately understood. This is the traditional path that science has so successfully followed since the Renaissance... In each of these cases, scientists built their theories upon a scaffold of experimental data. The Standard Model could not have been invented by theorists, however brilliant, just sitting around and thinking.

Sometimes scientists have followed a different road. The high road tries to avoid the morass of mundane experimental data. . .

Glashow goes on portraying the invention by Einstein of classical general relativity as the single example of successful pursuit of the high road; and exemplifying modern high-rollers with superstring theorists.

However, we ought to say, string theory in general is a very reasonable bet compared with most “quantum gravity” schemes. What motivates them? From a textbook [2, p. 24] we quote Bergmann:

Today’s theoretical physics is largely built on two giant conceptual structures: quantum theory and general relativity. As the former governs primarily the atomic and subatomic worlds, whereas the latter’s principal applications so far have been in astronomy and cosmology, our failure to harmonize quanta and gravitation has not yet stifled progress in either front. Nevertheless, the possibility that there might be some deep dissonance has caused physicists an esthetic unease, and it has caused a number of people to explore avenues that might lead to a quantum theory of gravitation, no matter how many decades away the observations. . .

Dissonance, we claim, there is not: trees electromagnetically keep growing on the third planet from the Sun, bound by gravity since as far as we can tell. There is theoretical ignorance about a vast region of possible experience unconstrained by evidence. Be that as it may, “esthetic unease” is about the worst guide for science. Ugliness is in the eye of the beholder. Nobody claims the standard model of particle physics to be beautiful. However, it has survived more than 35 years of determined theoretical and —much more important— empirical assault. It possesses now the beauty of staying power: any scheme whatsoever aiming to replace it needs to manage the Standard Model disguise.

History is a better guide. The clash between classical mechanics and electromagnetism, seemingly leading to catastrophic atomic collapse, was overcome by more profound experiments and the quantum theories designed to explain them. Therefore we do little of the “dissonance” of the underpinnings of quantum theory and classical gravity, since in all likelihood at least one of those is doomed to perish.

Glashow concludes:

History is on our side (i.e., of the low-rollers). Every few years there has been a world-shaking new discovery in fundamental physics or cosmology... Can anyone really believe that nature’s bag of tricks has run out? Have we finally reached the point where there is no longer... a bewildering new phenomenon to observe? Of course not.

Fortunately, even classical gravity is in deep crisis. This opens a number of opportunities. The crisis concerns almost every aspect.

- Cosmic acceleration. In a nutshell, the expansion of the universe seems to be *accelerating* when it should be *braking*. This is the “cosmological constant” or “dark energy” problem. The question is obviously: why now? We shall come back to this.
- Galaxy clustering and cosmology. As it turns out, some think the previous to be a pseudo-problem. Wiltshire and coworkers [3, 4, 5, 6] have argued that:

Cosmic acceleration can be understood as an apparent effect, and dark energy as a misidentification of those aspects of cosmological gravitational energy that by virtue of the strong equivalence principle cannot be localized...

Wiltshire’s proposal is of the “radically conservative” kind. The implication is that we truly do not know how to solve the Einstein equations.

In a similar vein, current orthodoxy regarding gravitational collapse towards black holes and the “information loss” problem has been also called into question [7].

- The best-tested aspects of the theory are challenged by the Solar System anomalies. To begin with, at least since the eighties it has been known that the trajectories of the *Pioneer* 10 and *Pioneer* 11 past the outer planets’ orbits deviate from the predictions, as though some extra force is tugging at them from the direction of the Sun [8, 9, 10].

The unmodeled blue shift appearing in the *Pioneer* missions data amounts to  $10^{-9}$  cm s<sup>-2</sup>; it may not seem much, but it adds now to many thousands of kilometres behind the projected paths. A “covariant” solution to the anomaly seems ruled out —see for instance [11]. In desperation, some bold proposals are being made. For instance that, because of the influence of background gravitational sources in the universe on the evolving quantum vacuum [12, 13], astronomical time and time as nowadays measured by atomic clocks might not coincide.

- To this, add the even more surprising and now apparently verified fact (spoken about in hushed tones since 1990, when first noticed in the flight of probe *Galileo* by Earth), that the slingshot manoeuvre of spacecraft delivers (or takes away) more energy than the current theory allow us to expect [14]. A simple empirical formula describes rather accurately the deviations, which translate into a few millimetres a second of extra velocity.

Both solar system anomalies belong in the category of “unexpected experiments”.

- The existence of (non-baryonic) *dark matter* is better established than that of dark energy, since several lines of evidence point to a relatively low baryon content of the universe.

However, models do exist that attribute the relatively high acceleration of stars in a typical galaxy, thus the appearance of dark matter, to mysterious deviations from standard gravity. Particularly, Milgrom’s MOND (modified Newton dynamics) model —see [15] and references therein, as well as the discussion in the popularization book [16]. MOND postulates that Newton’s law is modified in very weak acceleration regimes. There is no “respectable” theory behind it as yet. However, as it happens, Milgrom’s hypothesis implies predictions on the surface densities of galaxies and more; these have been pretty much verified till now. The Milgrom acceleration is pretty close to the cosmic acceleration. It is not very different in order of magnitude from the “acceleration” of the *Pioneers*. On the other hand, interaction with dark matter might explain the *Pioneers*’ blue shift.

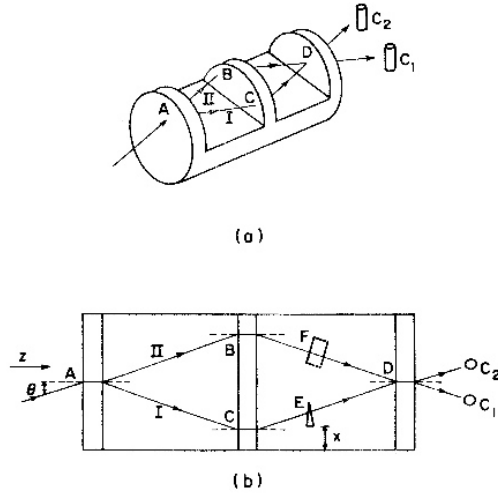
- Taken together, dark matter and energy signal the transition to a new cosmological paradigm. Whether they will emerge as modified gravity (massive graviton or other), new energy components, or pointers to strings and other noncommutative substructures, remains to be seen.
- Among the questions of principle that periodically erupt into controversy, is the question of the speed of transmission of the gravitational interaction, or, if you wish, the lack of aberration of gravity [17].

## 2 Gravity and experiment: expect the unexpected

Perhaps the most fundamental question of principle, for our purposes, concerns the role, if any, of the principle of equivalence in the interface of gravity with the quantum world. We begin by that in earnest. Now, there is little in the way of quantum gravity that we can probe in laboratory benches at present. The universe was created with a quarantine: gravity is so weak an interaction that it can only produce measurable effects in the presence of big masses, and this very fact militates against detecting radiative corrections to it. To see quantum effects in pure gravity is far beyond our power. What we can do with some confidence is to envisage quantum systems in classical background gravitational fields, with back-reaction neglected, or approximately treated. In fact, only the interface of nonrelativistic quantum mechanics with Newtonian gravity has been experimentally tested.

Some wisdom is gained, however, by not discarding a priori such humble beginnings. For this writer, the alpha of quantum gravity is the Colella–Overhauser–Werner (COW for short, from now on) experiment [18]. It tests the equivalence principle. The latter appears in textbooks in slightly different formulations. For some, the “strong” principle says that accelerative and gravitational effects are locally equivalent; the “weak” principle states that inertial masses and gravitational charges are the same (up to a universal constant). Some others use the nomenclature the other way around. In both cases we refer to systems placed in external fields, such that the complicating effects of the gravitational pull by the system itself can be neglected. From the second form it plainly follows that all *classical* masses fall

**Fig. 1** (a) In the most common interferometer three “ears” are cut from a perfect crystal, ensuring coherence over it (about 10 cm long). The incident beam is split (by Bragg scattering) at *A* into two, I and II. These are redirected at *B* and *C* and recombine in the last ear. The relative phase at *D* determines the counting rate at the detectors. (b) Top view of the interferometer. The relative phase can be changed in a known way by inserting a wedge in one beam at *E*, which thickness can be changed by displacement. The experiment is performed at *F*.



with the same acceleration in a gravity field. Thus, if the initial conditions for those masses coincide, their trajectories will coincide as well: Galileo’s uniqueness of free fall. In other words, mass is superfluous to describe particle motions in classical gravity; it all belongs to the realm of kinematics. From this to the assertion [19, p. 334] that

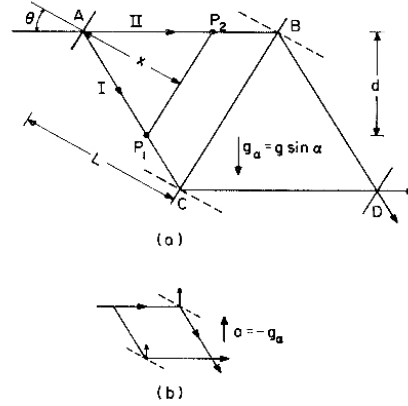
... geometry and gravitation were one and the same thing.

is there but a near-vanishing step.

So, what does the COW experiment mean for humanity? It and its follow-ups lend support to the equivalence principle. It would have been earth-shaking if they did not; but it is indispensable to reflect on which aspects of current orthodoxy are confirmed, and which ones actually disproved by it.

The COW tool is neutron (and neutral atom) interferometry. A typical neutron interferometer —see Fig. 1, taken from [20]— is a silicon crystal of length  $L$ . The incident beam is split with half-angle  $\theta$  in the first ear of the apparatus at one extreme, redirected halfway through it, and recombines in the third ear at the other extreme. The neutron wavelength  $\lambda_N$  and the atom spacing in the crystal need to be of the same order, about  $10^{-8}$  cm. Thus the momentum is in the ballpark of  $(\hbar/\lambda_N) \sim 10^{-20}$  erg. The neutron is relatively cold: with an inertial mass  $m_i \sim 10^{-24}$  g, this implies a velocity  $v \sim 10^4$  cm/s; thus a nonrelativistic calculation will do.

A gravitational phase shift is obtained simply by rotating the apparatus about the incident beam, say an angle  $\alpha$ , so the acceleration is  $g \sin \alpha$ , with  $g$  the standard acceleration on Earth. The phase shift over one period is of the order of the quotient between the (difference in) potential energy and the kinetic energy of the beam; even with the small velocities involved, this is of the order  $\sim 10^{-7}$ . Under such conditions, it is not hard to see that the phase difference is given approximately by



**Fig. 2** Gravitational perturbation of the beam. (a) The interferometer is rotated around the incident beam by an angle  $\alpha$ ; the beams will be at a different height (equal to  $2x \sin \theta$  between equivalent points along the paths), with an effective gravitational field  $g_\alpha = g \sin \alpha$  in the interferometer plane. (b) In the free-fall system, the neutrons beam are unaccelerated, but the interferometer scattering planes appear to be accelerating upwards.

$$\frac{\int V dt}{\hbar},$$

where  $V$  denotes the difference in potential between the higher and the lower unperturbed neutron paths and  $t$  is the time.

Now, let  $x$  be a rectilinear coordinate along the long diagonal of the rhomb constituted by the two beam's paths. Then the difference of height between the paths is as indicated in Fig. 2. The difference in potential is  $2mg \sin \alpha x \sin \theta$ . Thus we have:

$$\frac{\int V dt}{\hbar} = \frac{4mg \sin \alpha \sin \theta}{\hbar v \cos \theta} \int_0^L x dx = \frac{mgA \sin \alpha}{\hbar v}, \quad (1)$$

with  $v$  the mean velocity of the neutrons and  $A$  the area of the rhomb, given by half the diagonals' product:

$$A = 2L^2 \tan \theta.$$

Actually the mass appearing in (1) is the gravitational charge; the inertial mass  $m_i$  is hidden in the relation between  $v$  and the de Broglie wavelength. The shift (1) is around 100 rad, and the resulting fringe pattern easily visible and measurable. (We have neglected the effect of the Earth's rotation, which amounts to less of 2% of the total shift.) It turned out that the neutrons do fall in the Earth's gravity field as predicted by the Schrödinger equation, with  $m$  and  $m_i$  identified.

The experiment appears to confirm both versions of the equivalence principle, since the possibility of describing the problem in the neutron beam reference system as an upward acceleration of the interferometer holds in the Schrödinger equation. This is discussed exhaustively in [21]. Use of the Dirac equation instead makes no

practical difference. Anyway, the experiment was repeated in “actually accelerated” interferometers, with the expected result [22].

However, as soon as we try to translate the “weak” principle in *geometrical* terms in the quantum context, we run into trouble. The fact that “trajectories” have not much quantum-mechanical meaning is enough to make us suspicious. Nevertheless, let us for simplicity explore the situation in terms of circular Bohr orbits. (That these are still pertinent concepts is plain to anybody who has done atomic physics with the Wigner phase-space function [23, 24].) Assume a very large mass  $M$  bounds a small one  $m$  gravitationally into a Bohr atom. For circular orbits with angular velocity  $\omega$ , Kepler’s laws give

$$\omega^2 = \frac{GM}{r^3}, \quad \text{with } r \text{ restricted by } mr^2\omega = n\hbar.$$

Thus

$$E_n = -\frac{1}{2}m\omega^2 r^2 = -\frac{G^2 M^2 m^3}{2\hbar^2 n^2}.$$

Therefore in quantum mechanics one can *tell the mass* of a gravitational bound particle. The explanation for this lies in the very quantization rule

$$[x, p] = i\hbar,$$

which is formulated in phase space. If we define velocity by  $p/m$ , we obtain the commutator

$$[x, v] = i\hbar/m.$$

This means that kinematical quantities are functions of  $\hbar/m$ . In general, it is enough to look at the Schrödinger equation to see that energy eigenvalues go like  $m f(\hbar/m)$ , or more accurately,  $m f(\hbar^2/mm_i)$  for some function  $f$ .

Now, if we admit the previous, how does the dependence of the mass disappear in the classical limit? The only possibility is that the quantum number scales with  $m$ . This of course makes sense in the semiclassical limit: if particle 1 is heavier than particle 2, we expect its energy levels to be accordingly higher. But for low-lying states geometrical equivalence inevitably breaks down. We have here the curious case of a symmetry generated (rather than broken) by “dequantization”. The point was made in [20].

In summary, lofty gravity is treated by quantum mechanics as lightly as lowly electrodynamics. In the classical motion of charged particles, only the parameter  $e/m$  appears. This is not interpreted geometrically, since  $e/m$  varies from system to system, so nobody thinks it has fundamental significance. When the system is quantized,  $\hbar$  comes along in both cases, and in gravity experiments, like the ones described above with states in the continuum, we can tell the mass. Alas, for some this destroys the beauty of the theory. So much that they never mention the fact.

## 2.1 Noncommutative geometry I

Before examining the consequences of the failure of the geometrical principle, let us see if we can find a way out. To preserve weak equivalence as an exact quantum symmetry, we must take the canonical velocity as a dynamical quantity  $\mathfrak{v}$ . Then the Hamiltonian is rewritten

$$H = m(\mathfrak{v}^2/2 + V(x)) = m\mathcal{H}(x, \mathfrak{v}),$$

with  $V$  the gravitational potential. If now we quantize the theory in terms of  $x$  and  $\mathfrak{v}$ , we obtain a “quantum gravity” theory respecting the geometrical equivalence principle (although, of course, this flies in the face of the workings of ordinary quantization for other interactions).

Through existence of the constant  $c$  of nature, such a quantization method involves the introduction of a fundamental length

$$[x, \mathfrak{v}] = icl_0.$$

This is not quite “noncommutative geometry” in the superficial way it is mostly practised nowadays (the present author is not innocent of such a sin), but resembles it more than a bit. The point we are able to make is twofold: (i) of need the geometrical approach to quantum gravity will be *noncommutative* or will not be; (ii) it is not at all required that  $l_0$  be of the order of Planck’s length scale. It has been argued many times, invoking mini-black holes in relation with the incertitude principle and such, that something must happen at that length scale —see [25] for example. But nothing forbids that the critical length be bigger (a string length, for instance), provided it could have escaped detection so far. If and how such fundamental length intervenes is a matter only for experiment to decide.

We return to noncommutative geometry in Section 5.

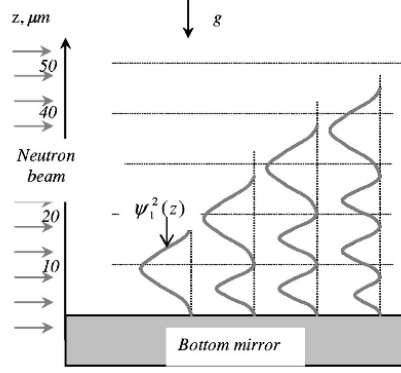
## 2.2 Whereto diffeomorphism invariance?

The understanding that geometry and gravitation are not to be one and the same thing should be confirmed by some experiment checking (low-lying) states of a quantum system bound by gravity.

Such an experiment —the first ever to observe gravitational quanta— has already taken place [26].

Ultracold neutrons ( $v \sim 10$  m/s) are stored in a horizontal vacuum chamber; a mirror is placed below and a non-specular scatterer above. Thus the neutrons find themselves in a sort of gravitational potential well, with a “soft wall” on one side. The Bohr–Sommerfeld formula is good enough to calculate its energy levels associated to vertical motion:





**Fig. 3** Quantum states are formed in the “potential well” between the Earth’s gravity field and the horizontal mirror on bottom. The vertical axis  $z$  is intended to give an idea about the spatial scale for the phenomenon.

$$E_n = (9m_N/8)^{1/3} (\pi\hbar g [n - \frac{1}{4}])^{2/3}.$$

We obtain

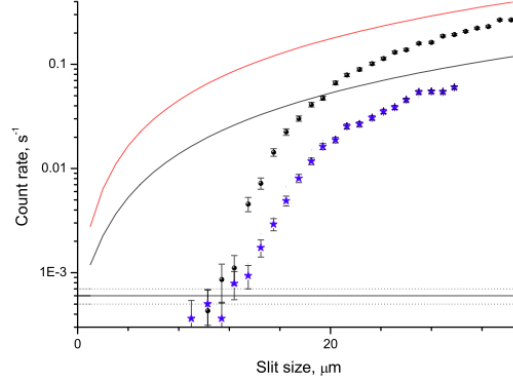
$$E_1 \simeq 1.4 \text{ peV} \simeq 10^{-13} \text{ Ry.} \quad (2)$$

A first remarkable thing is the minuteness of (2). In spite of being so small, quantum effects of gravity have been detected on a table-top! However, the main question here is that the difference between masses becomes of a yes/no nature. Suppose that the height of the “slit” formed by the upper and lower walls of the chamber is smaller than  $10^{-3}$  cm. If instead of neutrons one were trying to send through (say) aluminium atoms, they would be observed at the exit. However, that same slit *on Earth* is opaque to neutrons. The following rule of thumb is useful: the energy required to lift a neutron by  $10^{-3}$  cm is classically 1 peV with a good approximation. Accordingly the width of the state (2) can be estimated: the height of the chamber should be bigger than  $1.4 \times 10^{-3}$  cm for neutrons to be observed at the exit. Fig. 3 illustrates this. The phenomenon has nothing to do with diffraction, since the wavelength of neutrons remains much smaller than the height of the slit; visible light, with a wavelength much bigger than those neutrons, is transmitted.

Bingo! A slit has become a wall, impenetrable. Uniqueness of free fall fails. Gravitation is not just geometry.

The point is even more forcefully brought home in Fig. 4, which describes the actual experimental situation. Put in a different way, at least for interaction with matter, the (geometrical form of the) equivalence principle and the incertitude principle clash. No prizes to guess which must give way.

Surprisingly, our viewpoint is found controversial by some. To put matters into perspective, it is helpful to keep in mind that the equivalence principle is classically expressed by the statements (1) Gravitational mass equals inertial mass or (2) The



**Fig. 4** Dependency of the particle flux on the slit size. The circles indicate the experimental results [26] for a beam with an average value of 6.5 m/s for the horizontal velocity component. The stars show the analogous measurement with 4.9 m/s. The solid lines correspond to the classical expectation values for these two experiments. The horizontal lines indicate the incertitude in the detector background.

motion of particles in a gravity field is indifferent to their mass. While the COW experiment confirms (1), the second is untrue in the quantum world. Since point particles, paths and clocks play an apparently essential role in the foundations of general relativity (see the remarks further below), and since it is hard to see how geometry could have come to such a preponderance in dynamics without (2), it would seem the latter is bound to diminish. However, one can argue for an important residual role of geometry in quantum physics, as in the very readable article [27].

(In the current experimental situation, there is not much more than can be done directly to measure quantum jumps in a gravitational field. Present hopes to improve on accuracy of measurement of the quantum states parameters rest on use of storage sources of ultra-cold neutrons and magnetic field gradients to resonate with the frequency defined by the energy difference of two states [28].)

Among the numerous works on “quantum gravity” that make much of the classical geometry aspects of gravitation, a good representative is the homonymous book [29]. Its philosophical position is staked out at the outset:

... the question we have to ask is: what we have learned about the world from quantum mechanics and from general relativity?... What we need is a conceptual scheme in which the insights obtained with general relativity and quantum mechanics fit together.

This view is not the majority view in theoretical physics, at present. There is consensus that quantum mechanics has been a conceptual revolution, but many do not view general relativity in the same way... According to this opinion, general relativity should not be taken too seriously as a guidance for theoretical developments.

I think that this opinion derives from a confusion: the confusion between the specific form of the Einstein–Hilbert (EH) action and the modification of the notions of space and time engendered by general relativity.

We are pleased to vote with the bread-and butter majority here. The trouble is the non-geometrical cast of quantum *dynamics*. Since we know not the shape of things to come, the task is not so much to “fit general relativity with quantum mechanics together” as to —slowly and painstakingly— extend our knowledge to quantum and gravitational phenomena simultaneously taking place. It is somewhat saddening that the COW experiment and its successors are not found in the reference list of [29]; nor are they mentioned in the history of quantum gravity given as an appendix in that book —which is more in the “history of ideas” mold. In fact the sphere of ideas around the proper interpretation of the COW experiment hails back to Wigner, who, long ago, had explained keenly the quantum limitations of the *concepts* of general relativity [30], concluding:

... the essentially non-microscopic nature of the general relativistic concepts seems to us inescapable.

In otherwise mathematically subtle and full of gems [29], as in the works of other practitioners of quantum gravity, the warning goes unmentioned, as well as unheeded.

To summarize, a generous dose of salt is in order when dealing with “quantum gravity” claims. Without necessarily enjoying the quarantine, we should go most carefully about breaking it. Not only “large fragments of the physics community”, but also thoughtful mathematicians like Yuri Manin, advise a useful skepticism, in the respect of taking as physical what is just product of mathematical skill:

Well-founded applied mathematics generates prestige which is inappropriately generalized to support quite different applications. The clarity and precision of mathematical derivations here are in sharp contrast to the uncertainty of the underlying relations assumed. In fact, similarity of the mathematical formalism involved tends to mask the differences in the differences in the scientific extra-mathematical status... mathematization cannot introduce rationality in a system where it is absent... or compensate for a deficit of knowledge.

This as very timely quoted in [31].

### 3 Gravity from gauge invariance in field theory

From our standpoint, the action for gravitational interactions is more important than speculative “background independency” in a “final unified theory”. Moreover, the pure gravity EH action can be rigorously derived from the theory of quantum fields: a simple lesson, often forgotten. We proceed to that in this section. (As a historical note, for once the Einstein–Hilbert surname is right on the mark: independently Hilbert and Einstein gave the new equations of gravitation in the dying days of November 1915.)

### 3.1 Preliminary remarks

The book [32], containing lectures by Feynman on gravitation given at Caltech in 1962-63, deals with the *perturbative* approach to classical gravity; to wit, with the self-consistent theory of a massless spin-2 field (we may call it graviton). The foreword of this book (by John Preskill and Kip S. Thorne) is recommended reading. There the unfolding of (earlier) variants of the same idea by Kraichnan and Gupta is narrated as well, with references to the original literature. The main aspect in Kraichnan–Gupta–Feynman arguments is that a geometrical theory is obtained from flat-spacetime physics by using consistency requirements. Later work by Deser and Ogiwetsky and Polubarinov in the same spirit is also remarkable.

The distinctively non-geometrical flavour is welcome here, where we regard the geometrical approach as suspect. An excellent review with references of the classical path from the action for such field to the EH action is found in the recent book [33, Chap. 3].

Weinberg’s viewpoint in 1964 [34] is also very instructive and deserves mention. On the basis of properties of the  $\mathbb{S}$ -matrix, he proves that gravitons must couple to all forms of energy in the same way. He moreover shows that any particle with inertial mass  $m_i$  and energy  $E$  has, apart from Newton’s constant, an effective gravitational charge

$$2E - m_i^2/E.$$

For  $E = m_i$ , one recovers the usual equivalence result. While for  $m_i = 0$  one obtains  $2E$ , which gives the correct result for the deflection of light. (Also, a graviton must respond to an external gravity field with the same charge.)

In this section we perform a parallel exercise to Feynman’s: assuming ignorance of Einstein’s general relativity, we arrive again at the EH action by successive approximation. Our method has little to do with the “effective Lagrangians” approach and differs from traditional ones mentioned above in at least one of several respects:

- We consider only pure gravity. Coupling to matter is sketched after the fact, just for completeness.
- It is fully quantum field theoretical, in that recruits the *canonical formalism* on Fock space and quantum gauge invariance. Our main tool is BRS technology, and ghost fields are introduced from the outset. In other words, we treat gravity as any other gauge theory in the quantum regime; we obtain a quantum theory of the gravitational field, in which at some point we put  $\hbar = 0$ .
- We use the causal (or Epstein–Glaser) renormalization scheme [35], relying on the (perturbative expansion in the coupling parameter of the)  $\mathbb{S}$ -matrix. This entails a slight change of interpretation, in regard to renormalization, with respect to standard thinking; we briefly discuss the matter at the end of subsection 3.6. Epstein–Glaser renormalization is specially appropriate for gravity issues since it does not rely on translation invariance.
- We never invoke the stress-energy tensor.

In some sense we close a circle opened as well by Feynman in the early sixties [36], where he first realized that unitarity at (one-)loop graph calculations demanded ghost fields, for gravity as well as for Yang–Mills theory. Through well-known work by DeWitt, Slavnov, Taylor, Fadeev and Popov, and Lee and Zinn-Justin, this would eventually lead to BRS symmetry by the mid-seventies.

We mainly follow [37, 38]. The remote precedent for the last paper is an outstanding old article by Kugo and Ojima [39].

### 3.2 *Exempli-gratiae*

In order to make clear the strategy, we briefly recall here the similar treatment for (massive and massless) electrodynamics. Suppose we wish to effect the quantization of spin-1 particles by means of real vector fields. The question is how to eliminate the unphysical degrees of freedom, since a vector field has four independent components, while a spin-1 particle has three helicity states, or two if it is massless.

A standard procedure is to impose the constraint  $\partial^\mu A_\mu =: (\partial \cdot A) = 0$ . However, this is known to lead to the Proca Lagrangian (density), which has very bad properties. Also, under quantization, use of Proca fields entails giving up covariant commutators of the disarmingly simple form found for neutral scalar fields:

$$[A^\mu(x), A^\nu(y)] = i\eta^{\mu\nu}D(x-y), \quad (A^\mu)^+ = A^\mu; \quad (3)$$

with  $\eta$  the Minkowski metric and  $D$  the Jordan–Pauli propagator. We would like to keep them instead. The Klein–Gordon equations

$$(\square + m^2)A^\mu = 0 \quad (4)$$

we would like to keep as well. Now, it is certainly impossible to realize (3) and (4) on Hilbert space if by  $+$  we understand the ordinary involution. However, it is possible to do it through the introduction of a distinguished symmetry  $\eta$  (that is, an operator both selfadjoint and unitary), called the Krein operator. Whenever such a Krein operator is considered, the  $\eta$ -conjugate  $O^+$  of an operator  $O$  with adjoint  $O^\dagger$  is:

$$O^+ := \eta O^\dagger \eta.$$

Let  $(\cdot, \cdot)$  denote the positive definite scalar product in  $H$ . Then

$$\langle \cdot, \cdot \rangle := (\cdot, \eta \cdot)$$

yields an “indefinite scalar product”, and the definition of  $O^+$  is just that of the adjoint with respect to  $\langle \cdot, \cdot \rangle$ . Then  $A$  will be self-conjugate.

The massive vector field model is known to be a gauge theory [40] if we introduce the auxiliary (scalar) Stückelberg field  $B$  (say with the same mass  $m$ ), and gauge transformations of the form:

$$\begin{aligned}\delta A^\mu(x) &= \eta^{\mu\nu} \partial_\nu \theta(x) = \partial^\mu \theta(x); \\ \delta B(x) &= m\theta(x).\end{aligned}$$

The trick now is to use the unphysical parts  $\partial \cdot A, B$  plus the ghosts  $u$  and anti-ghost  $\tilde{u}$  to construct the BRS operator

$$Q = \int_{x^0=\text{const}} d^3x (\partial \cdot A + mB) \overleftrightarrow{\partial}_0 u,$$

whose action should reproduce the gauge variations (where commutators  $[\cdot, \cdot]_-$  or anticommutators  $[\cdot, \cdot]_+$  are taken according to whether the ghost number of the varied field is even or odd):

$$\begin{aligned}sA^\mu(x) &= [Q, A^\mu(x)]_\pm = i\partial^\mu u(x); \\ sB(x) &= [Q, B(x)]_\pm = imu(x); \\ su(x) &= [Q, u(x)]_\pm = 0; \\ s\tilde{u}(x) &= [Q, \tilde{u}(x)]_\pm = -i(\partial \cdot A(x) + mB(x)).\end{aligned}\tag{5}$$

With these relations one easily proves 2-nilpotency modulo the field equation:

$$2Q^2 = i \int_{x^0=\text{const}} d^3x \square u \overleftrightarrow{\partial}_0 u + im^2 \int_{x^0=\text{const}} d^3x u \overleftrightarrow{\partial}_0 u = 0.$$

Thus the right hand side of (5) are coboundary fields. With the help of nilpotency, the finite gauge variations for the same fields of (5) are easily computed. The supercharge  $Q$  is conserved. The massless limit is not singular in this formalism: for photons, we just put  $m = 0$ , and  $B$  drops out of the picture.

### 3.3 The free Lagrangian

A rank 2 tensor field under the Lorentz group decomposes into the direct sum of four irreducible representations, corresponding to traceless symmetric tensors, a scalar field, and self-dual and anti-self-dual tensors. We group the first two into a symmetric tensor field  $h \equiv \{h^{\mu\nu}\}$  with arbitrary trace. Let us introduce as well

$$\varphi := h^\rho_\rho; \quad H \equiv \{H^{\mu\nu}\} := \{h^{\mu\nu} - \frac{1}{4}\eta^{\mu\nu}\varphi\}; \quad \text{thus} \quad H^\rho_\rho = 0.$$

(We wish to keep  $h$  to denote the whole tensor, and so we do not use the standard notation for its trace.) Again the question is how to eliminate the superfluous degrees of freedom in the description of a spin-2 relativistic particle, which possesses only two helicity states. A fortiori we do not want to follow for the graviton the path of enforcing constraints, that was discarded for photons.

For a free graviton one may settle on the Lagrangian

$$\mathcal{L}^{(0)} = \frac{1}{2}(\partial_\rho h^{\alpha\beta})(\partial^\rho h_{\alpha\beta}) - (\partial_\rho h^{\alpha\beta})(\partial_\beta h_\alpha^\rho) - \frac{1}{4}(\partial_\rho \varphi)(\partial^\rho \varphi). \quad (6)$$

Of course this choice is not unique. The more general Lorentz-invariant action quadratic in the derivatives of  $h$  is of the form

$$\int d^4x [a(\partial_\rho h^{\alpha\beta})(\partial^\rho h_{\alpha\beta}) + b(\partial_\rho h^{\alpha\beta})(\partial_\beta h_\alpha^\rho) + c(\partial_\rho \varphi)(\partial^\sigma h_{\rho\sigma}) + d(\partial_\rho \varphi)(\partial^\rho \varphi)].$$

The frequently invoked Fierz–Pauli Lagrangian [41] is of this type, with  $a = \frac{1}{4}, b = -\frac{1}{2}, c = \frac{1}{2}, d = -\frac{1}{4}$ . The signs are conventionally chosen in both cases so that the first term has a positive coefficient. The Euler–Lagrange equations corresponding to (6):

$$\partial_\gamma \frac{\partial \mathcal{L}^{(0)}}{\partial (\partial_\gamma h_{\alpha\beta})} = 0$$

yield at once

$$\square h^{\alpha\beta} - \partial_\gamma \partial^\beta h^{\alpha\gamma} - \partial_\gamma \partial^\alpha h^{\beta\gamma} - \frac{1}{2} \eta^{\alpha\beta} \square \varphi = 0. \quad (7)$$

This form is essentially equivalent to the Fierz–Pauli equation, but more convenient here. (For a critique of the Fierz–Pauli framework, consult [42].)

### 3.4 A canonical setting

A crucial point is the invariance of the Lagrangian  $\mathcal{L}^{(0)}$ —thus of equation (7)—under gauge transformations

$$\delta h^{\alpha\beta} = \lambda(\partial^\alpha f^\beta + \partial^\beta f^\alpha - \eta^{\alpha\beta}(\partial \cdot f)) = \lambda b_\tau^{\alpha\beta\rho} \partial_\rho f^\tau, \quad (8)$$

where

$$b_\tau^{\alpha\beta\rho} := \eta^{\alpha\rho} \delta_\tau^\beta + \eta^{\beta\rho} \delta_\tau^\alpha - \eta^{\alpha\beta} \delta_\tau^\rho,$$

for arbitrary  $f = (f^\alpha)$ . This entails

$$\delta \varphi = -2\lambda(\partial \cdot f). \quad (9)$$

To verify this invariance, with an obvious notation, and up to total derivatives,

$$\begin{aligned} \delta \mathcal{L}_I^{(0)} &= -\delta h_{\alpha\beta} \square h^{\alpha\beta}; \\ \delta \mathcal{L}_{II}^{(0)} &= \delta h_{\alpha\beta} \partial^\rho (\partial^\alpha h_\rho^\beta + \partial^\beta h_\rho^\alpha); \\ \delta \mathcal{L}_{III}^{(0)} &= \frac{1}{2} \delta \varphi \square \varphi. \end{aligned}$$

One finishes the argument by use of (8) and (9).

That tensor  $b$  will reappear often. Classically, one could specify here the *transverse gauge* condition:

$$\partial_\beta (h^{\alpha\beta} + \delta h^{\alpha\beta}) = 0. \quad (10)$$

(In the gravity literature a so-called de Donder gauge condition is more frequently used.) The last equation is obtained at once if  $f^\alpha$  solves

$$\lambda \square f^\alpha = -\partial_\beta h^{\alpha\beta} =: -(\partial \cdot h)^\alpha;$$

then (7) reduces to  $\square h = 0$ .

As advertised, we refrain from quotient by imposing gauge conditions. In our BRS-like treatment, the elimination of the many extra degrees of freedom takes place cohomologically, rather than by use of constraints. The fields are promoted to (by now still free) normally ordered quantum fields. Clearly, in this approach we need to add to  $\mathcal{L}^{(0)}$  the gauge-fixing and free ghost terms:

$$\mathcal{L}_{\text{free}} = \mathcal{L}^{(0)} + \frac{1}{2}(\partial \cdot h) \cdot (\partial \cdot h) - \frac{1}{2}(\partial_\mu \tilde{u}_\nu + \partial_\nu \tilde{u}_\mu)(\partial^\mu u^\nu + \partial^\nu u^\mu - \eta^{\mu\nu}(\partial \cdot u)). \quad (11)$$

One quantizes  $h$  in the most natural way

$$[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = i b^{\alpha\beta\mu\nu} D(x-y); \quad (12)$$

and therefore the propagators for  $H, \varphi$  are given by:

$$\begin{aligned} [H^{\alpha\beta}(x), H^{\mu\nu}(y)] &= i(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}) D(x-y), \\ [\varphi(x), \varphi(y)] &= -8iD(x-y), \\ [\varphi(x), H^{\mu\nu}(y)] &= 0. \end{aligned}$$

Also, for the fermionic ghosts we have the anticommutation relations

$$[u^\alpha(x), u^\beta(y)] = i g^{\alpha\beta} D(x-y) \quad (13)$$

All other anticommutators vanish. The new Euler–Lagrange equations give rise now to the simplest possible, ordinary wave equations for all fields considered.

$$\square h = 0; \quad \square u = 0; \quad \square \tilde{u} = 0.$$

We can prove directly consistency of rules (12) and (13), analogous to (3) and (4), by constructing an explicit representation in a Fock–Krein space. The reader will see this in a later subsection.

Let us now introduce the BRS operator

$$Q = \int_{x^0=\text{const}} d^3x (\partial \cdot h)^\alpha \overleftrightarrow{\partial}_0 u_\alpha = \int_{x^0=\text{const}} d^3x ((\partial \cdot H)^\alpha + \frac{1}{4}\partial^\alpha \varphi) \overleftrightarrow{\partial}_0 u_\alpha; \quad (14)$$

where  $(\partial \cdot h)^\alpha$  denotes the divergence  $\partial_\beta h^{\alpha\beta}$ , which in view of (10) is unphysical, and  $u_\alpha$  is the fermionic (vector) ghost field. The associated gauge variations are:



$$\begin{aligned}
sh^{\mu\nu} &= [Q, h^{\mu\nu}] = ib_\tau^{\mu\nu\rho} \partial_\rho u^\tau = i(\partial^\mu u^\nu + \partial^\nu u^\mu - \eta^{\mu\nu}(\partial \cdot u)); \\
su &= [Q, u]_+ = 0; \\
s\tilde{u} &= [Q, \tilde{u}]_+ = -i(\partial \cdot h)^\mu.
\end{aligned} \tag{15}$$

Note that the action of the coboundary operator is dictated by the variation (8). Other important coboundaries like

$$s\varphi = i(\partial \cdot u); \quad s(\partial \cdot h)^\mu = 0$$

follow from (15) on-shell. Again the supercharge  $Q$  is 2-nilpotent and conserved.

### 3.5 What to expect

We make a temporary halt to examine whether, with our choices in subsection 3.3 we are on the right track, after all. Let  $g := (g_{\alpha\beta})$  denote the metric tensor and  $R$  the Ricci curvature. As hinted above, for this writer the EH action (with  $c=1$ , and without the “cosmological constant”)

$$S_{\text{EH}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\det g} R = -\frac{1}{16\pi G} \int d^4x \mathfrak{g}^{\mu\nu} R_{\mu\nu}.$$

constitutes the alpha and omega of gravitation theory. Here  $G$  is Newton’s constant, equal to  $\hbar/m_{\text{Planck}}^2$ . We recall

$$\begin{aligned}
\Gamma_{\beta\gamma}^\alpha &= \frac{1}{2} g^{\alpha\mu} (\partial_\gamma g_{\beta\mu} + \partial_\beta g_{\gamma\mu} - \partial_\mu g_{\beta\gamma}); \quad \text{thus} \\
\partial_\alpha g^{\mu\nu} &= -\Gamma_{\gamma\alpha}^\mu g^{\gamma\nu} - \Gamma_{\gamma\alpha}^\nu g^{\gamma\mu} \quad (\text{vanishing covariant derivative}); \\
R_{\mu\nu} &= \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\mu\nu}^\beta \Gamma_{\beta\alpha}^\alpha - \Gamma_{\mu\alpha}^\beta \Gamma_{\beta\nu}^\alpha; \\
R &= g^{\alpha\beta} R_{\alpha\beta}.
\end{aligned} \tag{16}$$

It is convenient to have a special notation for

$$\Gamma_\mu := \Gamma_{\mu\alpha}^\alpha = \frac{1}{2} g^{\alpha\gamma} \partial_\mu g_{\alpha\gamma} = \frac{\partial_\mu (\det g)}{2 \det g} = \partial_\mu \left( \log \sqrt{-\det g} \right).$$

We have employed that the minors of  $g_{\alpha\beta}$  in  $\det g$  are equal to  $\det g g^{\alpha\beta}$ . Finally, the Goldberg tensor 1-density

$$\mathfrak{g}^{\alpha\beta} := \sqrt{-\det g} g^{\alpha\beta}$$

is —quite canonically, according to [43, Sect. 2.1]— a hero of our story.

Let us define  $\lambda = 4\sqrt{2\pi G}$  (essentially the inverse of Planck’s mass, in natural units). Since our approach to  $S_{\text{EH}}$  is perturbative, we need to rewrite the corresponding Lagrangian  $\mathcal{L}_{\text{EH}}$  as a series in the coupling constant  $\lambda$ . An old trick in classical gravity —see for instance [44, Sect. 93]— is to split off a divergence from  $\mathcal{L}_{\text{EH}}$

by using

$$\begin{aligned}\mathfrak{g}^{\mu\nu}\partial_\alpha\Gamma_{\mu\nu}^\alpha &= \partial_\alpha(\mathfrak{g}^{\mu\nu}\Gamma_{\mu\nu}^\alpha) - \Gamma_{\mu\nu}^\alpha\partial_\alpha(\mathfrak{g}^{\mu\nu}); \\ \mathfrak{g}^{\mu\nu}\partial_\nu\Gamma_\mu &= \partial_\nu(\mathfrak{g}^{\mu\nu}\Gamma_\mu) - \Gamma_\mu\partial_\nu(\mathfrak{g}^{\mu\nu}).\end{aligned}$$

With the help of previous equations, one finds

$$\mathfrak{g}^{\alpha\beta}R_{\alpha\beta} = H - \partial_\gamma(\mathfrak{g}^{\mu\gamma}\Gamma_\mu - \mathfrak{g}^{\mu\nu}\Gamma_{\mu\nu}^\gamma) =: H - \partial^\gamma D_\gamma, \quad (17)$$

where

$$H = \mathfrak{g}^{\alpha\beta}(\Gamma_{\alpha\rho}^\gamma\Gamma_{\beta\gamma}^\rho - \Gamma_{\alpha\beta}^\rho\Gamma_\rho).$$

The key step in our identification comes now: to make the contact between quantum field theory and general relativity, we postulate

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \lambda h^{\mu\nu}. \quad (18)$$

Remark that do *not* assume  $h$  to be small in any sense. In (17) above we separate the part of the vector  $D$  containing negative powers of  $\lambda$ :

$$D_\gamma = \frac{1}{\lambda}(\frac{1}{2}\partial_\gamma\varphi + \partial^\rho h_{\gamma\rho}) + D_\gamma^{(0)}. \quad (19)$$

The inverse matrix  $\mathfrak{g}_{\mu\nu}$  with  $\mathfrak{g}^{\mu\rho}\mathfrak{g}_{\rho\nu} = \delta_\nu^\mu$  formally becomes a series

$$\mathfrak{g}_{\mu\nu} = \eta_{\mu\nu} - \lambda h_{\mu\nu} + \lambda^2 h_{\mu\gamma}h_\nu^\gamma - \lambda^3 h_{\mu\gamma}h_\tau^\gamma h_\nu^\tau + \dots \quad (20)$$

Substituting this expression in the new form of the action  $(2/\lambda^2)\int d^4x H$ , we obtain a series as well:

$$\mathcal{L} = \sum_0^\infty \lambda^n \mathcal{L}^{(n)}. \quad (21)$$

(Actually, the Neumann series (20) is somewhat suspect, in view of convergence problems and other technical difficulties. One could see the Cayley–Hamilton theorem to obtain an exact expression for  $(\mathfrak{g}_{\mu\nu})$ .) The lowest order, at any rate, is indeed of order  $\lambda^0$  in view of the two derivatives inside  $H$ ; and it is seen to coincide with the free model of subsection 3.3. For completeness and use later on, we also report the three-graviton and four-graviton couplings:

$$\begin{aligned}\mathcal{L}^{(1)} &= (-\frac{1}{4}\partial_\rho\varphi\partial_\sigma\varphi + \frac{1}{2}\partial_\rho h^{\alpha\beta}\partial_\sigma h_{\alpha\beta} + \partial_\gamma h_\rho^\alpha\partial_\alpha h_\sigma^\gamma)h^{\rho\sigma}; \\ \mathcal{L}^{(2)} &= -h_{\alpha\beta}h_\beta^\rho(\partial_\nu h^{\alpha\mu})(\partial_\mu h^{\beta\nu}) - \frac{1}{2}h_{\rho\sigma}h_\beta^\rho(\partial_\alpha h^{\rho\beta})(\partial_\alpha\varphi) \\ &\quad - \frac{1}{4}h_{\nu\mu}(\partial_\alpha h^{\nu\mu})h_{\sigma\rho}(\partial^\alpha h^{\sigma\rho}) + \frac{1}{2}h_{\nu\mu}(\partial_\alpha h^{\nu\mu})h^{\alpha\beta}(\partial_\beta\varphi) \\ &\quad + h_{\beta\rho}h_\sigma^\beta(\partial_\mu h^{\rho\alpha})(\partial^\mu h_\alpha^\sigma) - h_{\alpha\rho}(\partial_\mu h_\sigma^\rho)(\partial_\nu h^{\alpha\rho})h^{\mu\nu} \\ &\quad + \frac{1}{2}h_{\alpha\rho}h_{\beta\sigma}(\partial_\mu h^{\alpha\sigma})(\partial^\mu h^{\beta\rho}).\end{aligned} \quad (22)$$

### 3.6 Causal gauge invariance by brute force

Interacting fields in Epstein–Glaser formalism are made out of free fields. The starting point for the analysis is the functional  $\mathbb{S}$ -matrix in the Dyson representation under the form of a power series:

$$\mathbb{S}(g) = 1 + T = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n). \quad (23)$$

The theory is constructed basically by using causality and Poincaré invariance of the scattering matrix to determine the form of the time-ordered products  $T_n$ . Only those fields should appear in  $T_n$  that already are present in  $T_1$ . The adiabatic limit on the “coupling functions”  $g(x) \uparrow 1$  is supposedly taken afterwards.

Causal gauge invariance (CGI) is formulated by the fact that  $sT_n = [Q, T_n]_{\pm}$  must be a divergence, keeping in mind that  $T_n$  and  $T'_n$  are equivalent if they differ by coboundaries.

In particular, first-order CGI means

$$sT_1(x) = i(\partial \cdot T_{1/1})(x).$$

For  $T_1$ , let us try a general Ansatz containing cubic terms in the fields and leading to a renormalizable theory. At our disposal there are three field sets:  $h, u, \tilde{u}$ . The most general coupling with vanishing ghost number *without derivatives* is of the form

$$a\varphi^3 + b\varphi h_{\nu\mu} h^{\nu\mu} + c h_{\mu\nu} h_{\gamma}^{\nu\mu} h^{\gamma\mu} + (u \cdot \tilde{u})\varphi + e h_{\nu\mu} u^{\nu} \tilde{u}^{\mu}.$$

Correspondingly, with ghost number one since the action of the BRS operator increases ghost number by one, we can have (with an obvious simplified notation)

$$T_{1/1}^{\mu} = a' u^{\mu} \varphi^2 + b' u^{\mu} h \cdot h + c' (u \cdot h)^{\mu} \varphi + d' u^{\alpha} h_{\alpha\beta} h^{\beta\mu} + e' u (u \cdot \tilde{u}).$$

Forlorn hope. It must be:

$$s(\partial \cdot T_{1/1}) = 0.$$

This condition has only the trivial solution  $T_{1/1} = 0$ .

Since one cannot form scalars with one derivative, we are forced to consider cubic couplings with two derivatives. This is the root of “non-normalizability” (in Epstein–Glaser jargon) of gravitation. There are 12 possible combinations in  $T_1$  involving only  $h$  with two derivatives, and 21 combinations in  $T_1$  involving  $h, u, \tilde{u}$ , with two derivatives and zero total ghost-number. At the end of the day, one obtains  $T_1 = T_1^h + T_1^u$ , with  $T_1^h$  *uniquely* proportional to  $\mathcal{L}^{(1)}$  (modulo physically irrelevant divergences), and

$$T_1^u = a(-u^{\alpha}(\partial_{\beta}\tilde{u}_{\rho})\partial_{\alpha}h^{\beta\rho} + (\partial_{\beta}u^{\alpha}\partial_{\alpha}\tilde{u}_{\rho} - \partial_{\alpha}u^{\alpha}\partial_{\beta}\tilde{u}_{\rho} + \partial_{\rho}u^{\alpha}\partial_{\beta}\tilde{u}_{\alpha})h^{\beta\rho}).$$

The calculations are excruciatingly long, and of little interest. They, as well as the explicit expression of  $T_{1/1}$ , can be found in [37], to which we remit. By the way, had

we tried to use

$$g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}$$

instead of (18), then  $T_1^h$  turns out much more complicated—even after elimination of a host of divergence couplings.

More intrinsically interesting are the calculations of CGI at second order, also done in [37], which indeed reproduce  $\mathcal{L}^{(2)}$ . For the higher-order analysis, one needs some (rather minimal) familiarity with the Epstein–Glaser method to inductively renormalize (i.e., to define) the time-ordered products  $T_n$ , based on splitting of distributions. This requires use of antichronological products, corresponding to the expansion of the inverse  $\mathbb{S}$ -matrix. If we write the inverse power series:

$$\mathbb{S}^{-1}(g) = 1 + \sum_1^\infty \frac{1}{n!} \int d^4x_1 \dots \int d^4x_n \bar{T}_n(x_1, \dots, x_n) g(x_1) \dots g(x_n),$$

then we have  $\bar{T}_{|N|}(N) = \sum_{k=1}^n (-)^k \sum_{\sqcup_{j=1}^k I_j = N} T_{|I_1|}(I_1), \dots, T_{|I_k|}(I_k)$ , where the disjoint union is over (non-empty) blocks  $I_j$ . For instance, the second order term  $\bar{T}_2(x_1, x_2)$  in the expansion of  $\mathbb{S}^{-1}(g)$  is given by

$$\bar{T}_2(x_1, x_2) = -T_2(x_1, x_2) + T_1(x_1)T_1(x_2) + T_1(x_2)T_1(x_1).$$

The inductive step is performed using the totally advanced and totally retarded products. For instance, at the lower orders:

$$\begin{aligned} A_2(x_1, x_2) &= \bar{T}_1(x_1)T_1(x_2) + T_2(x_1, x_2) = T_2(x_1, x_2) - T_1(x_1)T_1(x_2); \\ R_2(x_1, x_2) &= T_1(x_2)\bar{T}_1(x_1) + T_2(x_1, x_2) = T_2(x_1, x_2) - T_1(x_2)T_1(x_1); \\ A_3(x_1, x_2, x_3) &= \bar{T}_1(x_1)T_2(x_2, x_3) + \bar{T}_1(x_2)T_2(x_1, x_3) + \bar{T}_2(x_1, x_2)T_1(x_3) \\ &\quad + T_3(x_1, x_2, x_3); \\ R_3(x_1, x_2, x_3) &= T_1(x_3)\bar{T}_2(x_1, x_2) + T_2(x_1, x_3)\bar{T}_1(x_2) + T_2(x_2, x_3)\bar{T}_2(x_1) \\ &\quad + T_3(x_1, x_2, x_3). \end{aligned} \tag{24}$$

By the induction hypothesis  $D_{n+1} := R_{n+1} - A_{n+1}$  depends only on known quantities. Moreover  $D_{n+1}$  has causal support. If we can find a way to extract its retarded or the advanced part, that is, to split  $D_{n+1}$ , then we can calculate  $T_{n+1}(x_1, \dots, x_{n+1})$ .

Consider then  $D_2(x, y) = [T_1(x), T_1(y)]$ , the first causal distribution to be split. We have thus

$$\begin{aligned} sD_2(x, y) &= [sT_1(x), T_1(y)] + [T_1(x), sT_1(y)] \\ &= i\partial_\mu^x [T_{1/1}^\mu(x), T_1(y)] + i\partial_\mu^y [T_1(x), T_{1/1}^\mu(y)]; \end{aligned} \tag{25}$$

so that  $D_2$  is gauge-invariant; and the issue is how to preserve gauge invariance in the renormalization or distribution splitting. That is, we must split  $D_2$  and the commutators—without the derivatives—in the previous equation; then gauge invariance:

$$sR_2(x, y) = i\partial_\mu^x R_{2/1}^\mu(x) + i\partial_\mu^y R_{2/2}^\mu(y)$$

can only be (and is) violated for  $x = y$ , that is, by derivative terms in  $\delta(x - y)$ . That is to say, if *local* renormalization terms  $N_2, N_{2/1}^\mu, N_{2/2}^\mu$  can be found in such a way that

$$s(R_2(x, y) + N_2(x, y)) = i\partial_\mu^x (R_{2/1}^\mu + N_{2/1}^\mu) + i\partial_\mu^y (R_{2/2}^\mu + N_{2/2}^\mu),$$

with an obvious notation, then CGI to second order holds.

When computing in practice, one is liable to find identities in distribution theory like

$$\begin{aligned} & \partial_\mu^x [A(x)B(y)\delta(x-y)] + \partial_\mu^y [A(y)B(x)\delta(x-y)] \\ &= \partial_\mu A(x)B(x)\delta(x-y) + A(x)\partial_\mu B(x)\delta(x-y) \end{aligned} \quad (26)$$

$$\begin{aligned} \text{and } & A(x)B(y)\partial_\mu^x \delta(x-y) + A(y)B(x)\partial_\mu^y \delta(x-y) \\ &= A(x)\partial_\mu B(x)\delta(x-y) - \partial_\mu A(x)B(x)\delta(x-y). \end{aligned} \quad (27)$$

We make the following observation: since

$$A(x)B(y)\delta(x-y) = A(x)B(x)\delta(x-y),$$

it must be

$$\partial_\mu^x (A(x)B(y)\delta(x-y)) = \partial_\mu^x (A(x)B(x)\delta(x-y));$$

which forces

$$B(y)\partial_\mu^x \delta(x-y) = B(x)\partial_\mu^x \delta(x-y) + \partial_\mu B(x)\delta(x-y). \quad (28)$$

We are able to prove both (26) and (27) from (28).

$$\begin{aligned} & \partial_\mu^x [A(x)B(y)\delta(x-y)] + \partial_\mu^y [A(y)B(x)\delta(x-y)] \\ &= \partial_\mu A(x)B(x)\delta(x-y) + A(x)B(y)\partial_\mu^x \delta(x-y) \\ &+ \partial_\mu A(x)B(x)\delta(x-y) - A(y)B(x)\partial_\mu^x \delta(x-y) \\ &= \partial_\mu A(x)B(x)\delta(x-y) + A(x)B(y)\partial_\mu^x \delta(x-y) \\ &- A(x)B(x)\partial_\mu^x \delta(x-y) = \partial_\mu A(x)B(x)\delta(x-y) + A(x)\partial_\mu B(x)\delta(x-y); \end{aligned}$$

where we have used (28) twice. Analogously,

$$\begin{aligned} & A(x)B(y)\partial_\mu^x \delta(x-y) + A(y)B(x)\partial_\mu^y \delta(x-y) = A(x)B(x)\partial_\mu^x \delta(x-y) \\ &+ A(x)\partial_\mu B(x)\delta(x-y) - A(y)B(x)\partial_\mu^x \delta(x-y) \\ &= A(x)\partial_\mu B(x)\delta(x-y) - \partial_\mu A(x)B(x)\delta(x-y), \end{aligned}$$

using (28) twice again.

Again after excruciatingly long calculations, by the sketched method one recovers the four-graviton couplings (22), plus terms with ghosts that we omit. Nev-

ertheless, the road seems barred in that, in order to rederive the EH Lagrangian, one would have to perform an infinite number of calculations. Put in another way, we could not ever finish ascertaining that the EH Lagrangian fulfils CGI. (In a (re)normalizable theory it would be enough to verify CGI till third order, but this is not the case here.) For the latter, a better way can be contrived, though. Leaving aside the question of uniqueness (in spite of “folk theorems”, uniqueness there is not: see Section 4), one can jump to the conclusion that  $\mathcal{L}_{\text{EH}}$  does satisfy CGI. In the next subsection, we describe a simple, short and rigorous argument for this.

Before pursuing, we take stock: a classical Lagrangian is extracted from a quantum theory because, for all computations, naturally starting at  $T_2$ , only *tree diagrams* are considered. *Par ce biais-ci* the limit  $\hbar \downarrow 0$  is taken. Of course, it is legitimate to perform the CGI analysis on graphs containing loops. In that way, the appropriate radiative corrections to  $S_{\text{EH}}$  are obtained; although this is not for the fainthearted. See [45] for the graviton self-energy; discrepancies between the coefficients of those corrections are still found in the literature. Anomalies are lurking there as well.

A last comment is in order: we have not tackled the matter of (re)normalizability of the theory, which in terms of the  $T_n$  is a bit involved. Suffice here to say that the conclusion is similar to that of standard arguments (on the basis of the dimensionality of  $G$ , for instance). It is true that in causal (re)normalization, there are no ultraviolet divergences as such. There is a problem of correct definition of distributions involved in the perturbative expansion of the  $\mathbb{S}$ -matrix. The price of a “non-normalizable” theory like Einstein’s is an infinite number of normalization constants in the process of that definition. This is not automatically so damning (also in regard of the discussion in the previous section), since perhaps they could be fixed by experiments, or have unobservable consequences. At any rate, the famous one-loop finiteness result by ’t Hooft and Veltman —consult for instance the discussion in [46, Sect. III]— means that, at next order in pure gravity, no (new normalization constants and thus no) new geometrical invariants are introduced: another rule of the godly quarantine.

### 3.7 CGI at all orders: going for it

We rely in the following on a theorem by Dütsch [38]: BRS invariance of a Lagrangian, depending only on the fields and their first derivatives and carrying non-negative powers of the couplings, implies local conservation of the BRS current. The latter implies CGI in the Heisenberg representation for tree graphs; and this result is kept in passing to time-ordered products. BRS invariance means precisely that the action of the BRS operator on the Lagrangian is a divergence, without use of the field equations. This admitted, the proof of CGI for the EH Lagrangian —modified like in formula (17)— by means of the BRS formulation of gravity by Kugo and Ojima [39] is simplicity itself.

In (our version of) that formulation, one keeps (18) and uses new gauge variations. The coboundary operator now is of the form

$$s = s_0 + \lambda s_1.$$

Here  $s_0$  acts exactly like  $s$  of (5) and

$$\begin{aligned} s_1 h^{\mu\nu} &= i(h^{\mu\rho} \delta_\tau^\nu + h^{\nu\rho} \delta_\tau^\mu) \partial_\rho u^\tau - i \partial_\tau (h^{\mu\nu}) u^\tau; \\ s_1 u &= -i(u \cdot \partial) u; \\ s_1 \tilde{u} &= 0. \end{aligned} \tag{29}$$

*Sotto voce* we are introducing here the Lie derivative of  $(g^{\mu\nu})$  with respect to the ghost vector field, thus diffeomorphism invariance. The new Lagrangian, complete with gauge-fixing and ghost terms, is:

$$\mathcal{L}_{\text{total}} = -H + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}} = -H + \frac{1}{2}(\partial \cdot h) \cdot (\partial \cdot h) + \frac{i}{2}(\partial_\nu \tilde{u}_\mu + \partial_\mu \tilde{u}_\nu) s h^{\mu\nu}.$$

Of course  $\mathcal{L}_{\text{total}}$  is not diffeomorphism-invariant. Compare (11). Note that

$$\mathcal{L}_{\text{gf}} = \mathcal{L}_{\text{gf}}^{(0)} = -\frac{1}{2}(s\tilde{u})^2,$$

while  $\mathcal{L}_{\text{ghost}}$  has terms of order  $\lambda$ . From this,

$$s^2 h = 0; \quad s^2 u = 0; \quad s^2 \tilde{u}_\mu = -\frac{\delta S_{\text{total}}}{\delta \tilde{u}^\mu},$$

vanishing on-shell. It is known that [39] that

$$s\mathcal{L}_{\text{EH}} = -i\lambda \partial \cdot (u\mathcal{L}_{\text{EH}}),$$

and since, with

$$F^\alpha := (\partial^\rho h_{\beta\rho}) s h^{\alpha\beta} \quad \text{we have} \quad s(\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}) = i\partial \cdot F,$$

it would seem that BRS invariance is checked, and we are done. Actually  $\mathcal{L}_{\text{EH}}$  does not fulfil the conditions of Dütsch’s theorem. However, we can use (17) and (19) to conclude. Indeed

$$-sH = -i\lambda \partial \cdot (u\mathcal{L}_{\text{EH}}) - i\partial \cdot (sD) + \frac{i}{\lambda} \partial \cdot (\square u - \partial(\partial \cdot u)).$$

The last vector is conserved, but the point is that it cancels the term of the form

$$\frac{1}{\lambda} s \left( \frac{1}{2} \partial_\gamma \varphi + \partial^\rho h_{\gamma\rho} \right),$$

in  $sD$ . Then

$$s(\mathcal{L}_{\text{total}}) = s(-H + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}) = -i\partial \cdot \left( \lambda u \mathcal{L}_{\text{EH}} + sD - F - \frac{\square u}{\lambda} + \frac{\partial(\partial \cdot u)}{\lambda} \right);$$

that is

$$s(\mathcal{L}_{\text{total}}) = \partial \cdot I \quad \text{with } I \text{ of the form } I = \sum_{k=0}^{\infty} \lambda^k I^{(k)},$$

and all is well. (The funny and revealing thing in all this is that the parts in  $1/\lambda^2$  and  $1/\lambda$  in the EH Lagrangian do not contribute to the equations of motion.)

It is instructive to compare the tensor and vector cases. In order to see the parallel, one ought to replace (the massless version of) formulae (5) by

$$\begin{aligned} sA_a^\mu(x) &= iD_{ab}^\mu u_b(x); \\ su_a(x) &= -\frac{i}{2} g f_{abc} u_b u_c; \\ s\tilde{u}_a(x) &= -i(\partial \cdot A_a(x)). \end{aligned}$$

Like there, it is plain that the action of the BRS operator increases ghost number by one. Here  $f_{abc}$  denotes the structure constants of a Yang–Mills model, and  $D$  is the corresponding covariant derivative.

### 3.8 Details on quantization and graviton helicities

The reader might be curious to know how the physical degrees of freedom emerge under our canonical recipe.

Let us treat ghosts first. Consider a family of absorption and emission operators  $c_a^\alpha(\mathbf{k})$  with  $a = 1, 2$  and standard anticommutators

$$[c_a^\alpha(\mathbf{k}), c_b^\beta(\mathbf{k}') ]_+ = \delta_{ab} \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{k}'),$$

defining a *bona fide* Fock space; with the definitions

$$\begin{aligned} u^\alpha(x) &= (2\pi)^{-3/2} \int d\mu(k) (e^{-ikx} c_2^\alpha(\mathbf{k}) - g^{\alpha\alpha} e^{ikx} c_1^\alpha(\mathbf{k})^\dagger), \\ \tilde{u}^\alpha(x) &= -(2\pi)^{-3/2} \int d\mu(k) (e^{-ikx} c_1^\alpha(\mathbf{k}) + g^{\alpha\alpha} e^{ikx} c_2^\alpha(\mathbf{k})^\dagger), \end{aligned} \quad (30)$$

where  $d\mu(k)$  is the usual Lorentz-invariant volume over the lightcone. There is a Krein operator on the ghost Fock space that allows for  $u$  being self-conjugate and  $\tilde{u}$  being skew-conjugate. This can be achieved by

$$c_1^i(\mathbf{k})^+ = c_2^i(\mathbf{k})^\dagger; \quad c_2^i(\mathbf{k})^+ = c_1^i(\mathbf{k})^\dagger; \quad c_1^0(\mathbf{k})^+ = -c_2^0(\mathbf{k})^\dagger; \quad c_2^0(\mathbf{k})^+ = -c_1^0(\mathbf{k})^\dagger,$$

with  $i = 1, 2, 3$ . Then formulae (30) are rewritten

$$\begin{aligned} u^\alpha(x) &= (2\pi)^{-3/2} \int d\mu(k) (e^{-ikx} c_2^\alpha(\mathbf{k}) + e^{ikx} c_2^\alpha(\mathbf{k})^+) = u^\alpha(x)^+, \\ \tilde{u}^\alpha(x) &= (2\pi)^{-3/2} \int d\mu(k) (-e^{-ikx} c_1^\alpha(\mathbf{k}) + e^{ikx} c_2^\alpha(\mathbf{k})^\dagger) = -\tilde{u}^\alpha(x)^+, \end{aligned} \quad (31)$$



From (30) or (31) we obtain for  $u, \tilde{u}$  the wave equations. *Covariant* anticommutation relations (13) also follow.

Note now

$$t^{\alpha\beta\mu\nu} := \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}) = t^{\mu\nu\alpha\beta}.$$

That is,

$$(t^{\mu\nu\alpha\beta}) = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & \begin{pmatrix} 3/4 & -1/4 \\ -1/4 & 3/4 \end{pmatrix} & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

on a  $(0,0), (j,j), (0,j), (j,l)$  block basis, with  $j, l = 1, 2, 3, j \neq l$ ; and in particular

$$T \equiv (t^{\mu\mu\alpha\alpha}) = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & 3/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 3/4 \end{pmatrix}$$

on the  $(0,0), (1,1), (2,2), (3,3)$  basis. Next we note that

$$T = MM^\dagger, \quad \text{with} \quad M = \begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 & -1/2 \end{pmatrix}.$$

Next we invoke operators defining a Fock space:

$$[b_{\alpha\beta}(\mathbf{k}), b_{\mu\nu}(\mathbf{k}')] = \frac{1}{2}(\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})\delta(\mathbf{k} - \mathbf{k}'),$$

with  $b_{\alpha\beta} = b_{\beta\alpha}$ . Define now operators  $a_{\alpha\beta}$ , with  $a_{\alpha\beta} = a_{\beta\alpha}$  as well, by  $a_{\alpha\beta} = b_{\alpha\beta}$  for  $\alpha \neq \beta$  and

$$a_{\alpha\alpha} = \sum_{\beta} M_{\alpha\beta} b_{\beta\beta}.$$

The rule

$$[a_{\alpha\beta}(\mathbf{k}), a_{\mu\nu}^\dagger(\mathbf{k}')] = g^{\alpha\alpha} g^{\beta\beta} t^{\alpha\beta\mu\nu} \delta(\mathbf{k} - \mathbf{k}')$$

follows.

The scalar field is now constructed in a way close to the standard one:

$$\varphi(x) = (2\pi)^{-3/2} \int d\mu(k) (e^{-ikx} a(\mathbf{k}) - e^{ikx} a^\dagger(\mathbf{k})), \quad (32)$$

where the (not Lorentz-covariant) operators  $a^\#$  satisfy

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = 4\delta(\mathbf{k} - \mathbf{k}').$$

The traceless sector is represented

$$H^{\alpha\beta}(x) = (2\pi)^{-3/2} \int d\mu(k) (e^{-ikx} a_{\alpha\beta}(\mathbf{k}) + g^{\alpha\alpha} g^{\beta\beta} t^{\alpha\beta\mu\nu} e^{ikx} a_{\alpha\beta}^\dagger(\mathbf{k})).$$

Now one can verify (12) painstakingly.

The last task in this subsection is to identify finally the physical degrees of freedom. For that, let us choose and fix  $k^\mu = (\omega, 0, 0, \omega)$ . One can verify that the only states not present in  $\mathcal{Q}$  (that is, belonging to the kernel of  $[\mathcal{Q}, \mathcal{Q}^\dagger]_+$ ) are

$$(b_{11} - b_{22})^\dagger |0\rangle \quad \text{and} \quad b_{12}^\dagger |0\rangle = b_{21}^\dagger |0\rangle.$$

They correspond to linear polarization states. Their complex combinations (circular polarization states) may be represented by matrices

$$\varepsilon_\pm := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which transform like

$$\varepsilon'_\pm = e^{\pm 2i\phi} \varepsilon_\pm$$

under a rotation of angle  $\phi$  about the direction of propagation. The reader can verify this by using the generator of rotations

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The two  $\pm 2$  helicity states have been thereby identified. These states satisfy

$$\varepsilon_\pm^{\mu\nu} k_\nu = 0. \quad (33)$$

These conditions are not Lorentz-invariant. Notice the associated gauge freedom

$$\varepsilon_\pm^{\mu\nu} \rightarrow \varepsilon_\pm^{\mu\nu} + k^\mu f^\nu + f^\mu k^\nu - \eta^{\mu\nu} (k \cdot f).$$

We may add

$$\varepsilon_{\pm\nu}^\nu = 0. \quad (34)$$

This five conditions (33) and (34) are also possible for a massive graviton —say  $k = (m, 0, 0, 0)$ . Thus they characterize the spin two case in general, with up to five degrees of freedom. Now, for  $k$  lightlike as above, let  $e^1, e^2$  denote two spacelike vectors orthogonal to  $k$  and mutually orthogonal, say  $(0, 1, 0, 0), (0, 0, 1, 0)$ . The tensors

$$(k_\mu k_\nu) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; (k_\mu e_\nu^1 + e_\mu^1 k_\nu) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; (k_\mu e_\nu^2 + e_\mu^2 k_\nu) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

verify (33) and (34) as well. They represent the three helicity states that disappear in the massless case.

### 3.9 Final remarks

- The geometrical form of general relativity, due to Einstein, is supremely elegant for some. However, the accompanying *interpretation* clashes with the one advocated here, based in the identification of the quanta of the gravitational field and more-or-less standard quantum field theory procedures; not to speak of table-top experiments. Since experiments probe gravity theory to very low orders in  $G, \hbar$ , one should keep an open mind, and welcome any consistent quantum theory perturbatively compatible with general relativity. As string theory promises to be.
- Coupling to matter. The graviton naturally couples to another symmetric tensor field:

$$T_1^{\text{matter}} = i\lambda A_{\alpha\beta\mu\nu} h^{\alpha\beta} T^{\mu\nu} \quad \text{with} \quad sT = 0.$$

Consideration that  $sT_1^{\text{matter}}$  must be a divergence leads at once to

$$\partial_\mu T^{\mu\nu} = 0;$$

just like it leads to charge conservation in quantum electrodynamics. Of course, the only conserved second-rank symmetric tensor in Poincaré-invariant field theory is the stress-energy tensor.

- Infrared freedom: in the Epstein–Glaser dispensation, vacuum diagrams, as any others, are ultraviolet-finite. Because of their high degree of singular order, however, we are assured that they are infrared finite. Therefore the vacuum is stable (no colour confinement or anything of the sort): a bonus for quantum gravity.
- The CGI formalism allows one can deal with massive gravity as well [47], although the shortcut in subsection 3.7 apparently is not available. At the price of introducing Stückelberg-like vector Bose ghosts, the massless limit of massive gravity is relatively smooth. Suggestively, a cosmological constant  $\Lambda = m^2/2$ , with  $m$  the graviton mass, ensues; one is reminded of Mach’s principle, as well. Note that the Fadeev–Popov approach to ghosts in quantum gravity is linked to existence of quasi-invariant measures on diffeomorphism groups [48].

### 3.10 Other ways

- Path-integral quantization faces the stark difficulty (rather, the impossibility) of “counting” four-dimensional manifolds [49]. A way around it may be “dynamical triangulation” —see [50] and in the same vein the recent [51].
- We cannot close the section without mentioning the promise of “asymptotic safety” in quantum gravity, developed by Reuter and coworkers. Consult [52], and references therein. There are intriguing results within this approach, pointing out to effective 2-dimensionality of spacetime at the Planck scale —which has been used by Connes, somewhat dubiously, to justify that the finite noncommutative geometry part in his reinterpretation of the standard model Lagrangian be of  $KO$ -dimension 6 [53]. While, at the other end of the scale, exceptionally good infrared behaviour could mimic both “dark matter” and “dark energy” behaviour.
- In relation with the discussion at the end of subsection 3.6, support for the idea that UV divergences in gravity are not so intractable has come recently from work by Kreimer [54].

## 4 The unimodular theories

A recent edition of a standard text about cosmology by a well-respected author [55] ends with a chapter on “Twenty controversies in cosmology today”. In the first one, about general relativity, he declares:

In fact it is theories without effective rivals that require the most vigilant testing.

Without contradicting this wisdom, let me point out that general relativity has some rivals which are too close for comfort. In order to grapple with them, let us go back to the fundamentals. We did omit the proof of that, for suitable variations of the metric  $(g_{\alpha\beta})$ , the Einstein *field equations* in vacuum

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = 0. \quad (35)$$

are equivalent to

$$\frac{\delta S_{\text{EH}}}{\delta g_{\alpha\beta}} = 0.$$

It is worthwhile to go through that routine here. Now

$$S_{\text{EH}} = -\frac{1}{16\pi G} \int d^4x \sqrt{-\det g} (R - 2\Lambda).$$

Clearly

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \left[ -(R - 2\Lambda) \frac{\delta \sqrt{-\det g}}{\delta g_{\alpha\beta}} \right] + \sqrt{-\det g} [R^{\alpha\beta} \delta g_{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta}],$$

where we take into account

$$R^{\alpha\beta} \delta g_{\alpha\beta} = -R_{\alpha\beta} \delta g^{\alpha\beta}, \quad \text{since} \quad \delta g^{\rho\sigma} g_{\sigma\epsilon} + g^{\rho\sigma} \delta g_{\sigma\epsilon} = 0.$$

Now,

$$\delta \sqrt{-\det g} = -\frac{1}{2\sqrt{-\det g}} \frac{\partial(-\det g)}{\partial g_{\alpha\beta}} \delta g_{\alpha\beta} = \frac{1}{2} \sqrt{-\det g} g^{\alpha\beta} \delta g_{\alpha\beta}.$$

It is easy to show that the last term in  $\delta S_{\text{EH}}$  does not contribute to the variation of the action. Therefore

$$\frac{\delta S_{\text{EH}}}{\delta g_{\alpha\beta}} = \frac{\sqrt{-\det g}}{16\pi G} (R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \Lambda g^{\alpha\beta});$$

hence (35).

It is apparent that life would be much simpler if  $\sqrt{-g}$  were not a dynamical quantity. This is suggested by Weinberg in his well-known review [56], in relation with the discussion in Section 6; the idea basically goes back to Einstein. Let us see what happens. First of all  $\Lambda$  seems to vanish from the picture. Second, since now the action has to be stationary only with respect to variations keeping  $\det g$  invariant, that is  $g^{\alpha\beta} \delta g_{\alpha\beta} = 0$ , one gathers the elegant

$$R_{\text{trace-free}}^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{4} g^{\alpha\beta} R = 0.$$

As it turns out, these are the Einstein equations again! The reason is that the contracted Bianchi identities

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2} \nabla^\alpha R, \quad \text{that is} \quad \nabla_\beta G^{\alpha\beta} = 0,$$

are still valid. They can be derived from  $R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu}$  and the uncontracted Bianchi identities:

$$\partial_\tau R_{\mu\nu\rho\sigma} + \partial_\sigma R_{\mu\nu\tau\rho} + \partial_\rho R_{\mu\nu\sigma\tau} = 0.$$

Therefore, by integration,

$$-R = G^\alpha_\alpha = -4\kappa; \quad \text{and then} \quad G^{\alpha\beta} + \kappa g^{\alpha\beta} = 0,$$

which is but (35) with  $\kappa$  replacing  $\Lambda$ . However, the interpretation has changed. The term in  $\Lambda$  in the action does not contribute anything (so the Minkowski space is a solution of the field equations even in the presence of such a term); and  $\kappa$  arises as an initial condition.

*Remark 1.* The discussion in this section is mainly pertinent in the presence of matter. If we define here the matter stress-energy tensor  $T \equiv (T^{\alpha\beta})$  by

$$\delta S_{\text{matter}} =: \frac{1}{2} \int d^4x \sqrt{-\det g} T^{\alpha\beta} \delta g_{\alpha\beta},$$

then varying  $S_{\text{matter}} + S_{\text{EH}}$  while keeping the determinant fixed results in

$$R^{\alpha\beta}_{\text{trace-free}} = 8\pi G T^{\alpha\beta}_{\text{trace-free}}.$$

Since the conservation law  $\nabla \cdot T = 0$  holds, we have now

$$R - 8\pi G T^\alpha_\alpha = 4\kappa,$$

and finally

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \kappa g^{\alpha\beta} = 8\pi G T^{\alpha\beta},$$

exactly the usual Einstein equations in the presence of a cosmological constant term plus matter, with the mentioned replacement of  $\Lambda$  by  $\kappa$ , and the attending change of interpretation.

It should be remarked that we are not implying that the classical action for gravitational physics is invariant only under coordinate transformations (“transverse diffeomorphisms”) that preserve the volume element. This is a stronger claim. Elegant justification for it is found in [57]. In accordance with the above, all known tests of general relativity probe equally the (several) unimodular theories. It has been argued that the matter-graviton coupling gives rise to inconsistencies when “strong” unimodularity holds [58]; but this objection we know not in relation with weak unimodularity. Only quantum effects would in principle allow tell it and general relativity apart [59] —after all the “measure” of the quantum functional integral for gravity is changed. Meanwhile, the interest of the unimodular theory is twofold: as indicated by Weinberg, it alleviates the cosmological constant problem (Section 6); moreover, it is natural from the current formulation of noncommutative manifold theory (subsection 5.9.2). From the viewpoint of the preceding section, the key question is how the unimodular theory is arrived at the  $\hbar \downarrow 0$  limit of a quantum theory of gravitons. We must leave the matter aside.

## 5 The noncommutative connection

### 5.1 Prolegomena

There is no general theory of noncommutative spaces. The practitioners’ tactics has been that of multiplying the examples, whereas trying to anchor the generalizations on the more solid ground of ordinary (measurable / topological / differentiable / Rie-

mannian. . . ) spaces. This is what we try to do here, within the limitations imposed by the knowledge of the speaker.

The first task is to learn to think of ordinary spaces in noncommutative terms. Arguably, this goes back to the Gelfand–Naïmark theorem (1943), establishing that the information on any locally compact Hausdorff topological space  $X$  is fully stored in the commutative algebra  $C(X)$  of continuous function over it, vanishing at  $\infty$ . This is a way to recognize the importance of  $C^*$ -algebras, and to think of them as locally compact Hausdorff noncommutative spaces. If we had just asked for the functions to be measurable and bounded, we would have been led to von Neumann algebras. Vector bundles are identified through their spaces of sections, which algebraically are projective modules of finite type over the algebra of functions associated to the base space —this is the Serre–Swan theorem (1962). In this way, we come to think of noncommutative vector bundles.

Under the influence of quantum physics, the general idea is then to forget about sets of points and obtain all information from classes of functions; e.g. open sets in  $X$  are replaced by ideals. The rules of the game would then seem to be: (1) find a way to express a mathematical category through algebraic conditions, and then: (2) relinquish commutativity. This works wonders in group theory, which is replaced by bialgebra theory, relinquishing (co)commutativity. However, that kind of generalization quickly runs into sands, for two reasons: (i) Some mathematical objects, like differentiable manifolds, and de Rham cohomology, are reluctant to direct noncommutative generalization. The same is true of Riemannian geometry; after all, all smooth manifolds are Riemann. (ii) Genuinely new “noncommutative phenomena” are missed.

For instance, in the second respect, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle  $\theta$ , is generally of this type. In such cases, when we examine the matter from the algebraic point of view, we are sometimes able to obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative.

One can situate the beginning of noncommutative geometry (NCG) in the 1980 paper by Connes, where the ‘noncommutative torus’  $T_\theta$  was studied [60]. Not only is this algebra able to answer the question mentioned above, but one can decide what are the smooth functions on this noncommutative space, what vector bundles and connections on  $T_\theta$  are and, decisively, how to construct a Dirac operator on it.

Even now, the importance of this early example in the development of the theory can hardly be underestimated. The noncommutative torus provides a simple but nontrivial example of *spectral triple*  $(A, H, D)$  —see further on for the notation— or ‘noncommutative spin manifold’, the algebraic apparatus with which Connes eventually managed to push aside the obstacles to the definition of noncommutative Riemannian manifolds. The Dirac equation naturally lives on spin manifolds, and these constitute the crucial paradigm, too, for Connes program of research (and unification) of mathematics.

The more advanced rules of the game would now seem to be: (1) Escape the difficulties “from above” by finding the algebraic means of describing a richer structure. If we reformulate algebraically what a spin manifold is, we can describe its de Rham cohomology, its Riemannian distance and like geometrical concepts, algebraically as well. Choice of a Dirac operator  $D$  means imposing a metric. However, there is the risk that the link to the commutative world is obscured. Therefore: (2) Make sure that the link is kept. In other words, prove that a noncommutative spin manifold is in fact a spin manifold in the everyday sense (!) when the underlying algebra is commutative. In point of fact, the second desideratum only received a definitive, satisfactory answer a few weeks ago.

## 5.2 Ironies of history

The following quotation of a popular book [61] provides a convenient rallying point.

When physicists talk about the importance of beauty and elegance in their theories, the Dirac equation is often what they have in mind. Its combination of great simplicity and surprising new ideas, together with its ability both to explain previously mysterious phenomena and predict new ones [spin], make it a paradigm for any mathematically inclined theorist.

Thus the irony is in that, first and foremost [61],

Mathematicians were much slower to appreciate the Dirac equation and it had little impact on mathematics at the time of its discovery. Unlike the case with the physicists, the equation did not immediately answer any questions that mathematicians had been thinking about.

The situation changed only *forty years* later, with the Atiyah–Singer theory of the index.

A second and minor irony is that, now that spin manifold theory is an established and respectable line of mathematical business, its community of practitioners seems mostly oblivious to the fact it underpins a whole new branch/paradigm/method of doing mathematics (although something is being done to fill up this gap).

Now come the *informal rules* for noncommutative geometers —rules which in any society insiders recognize as the most binding. These seem to be: (1) Keep close to physics, and in particular to quantum field theory. There is no doubt that Connes came to his ‘axioms’ for noncommutative manifolds by thinking of the Standard Model of particle physics as a noncommutative space. (2) Try to interpret and solve most problems conceivably related to noncommutative geometry by use of spectral triple theory. This of course is not to everyone’s taste, and a cynic could say: “Whoever is good with the hammer, thinks everything is a nail”; moreover it is of course literally impossible, as the mathematical world teems with virtual objects for which complete taxonomy is an impossible task. It has proved surprisingly rewarding, however.

A caveat about (2): there is an underlying layer of index theory and  $K$ -theory, which is a deep way of addressing quantization. But even there, when you need to



compute  $K$ -theoretic invariants, you are led back to smoother structures where you have more tools, like  $(A, H, D)$ .

### 5.2.1 A first conceptual star

Let us we imagine a star, with NCG in the centre, of subjects intimately related to it. This will include:

- Operator algebra theory
- $K$ -theory and index theory
- Hochschild and cyclic homology
- Bialgebras and Hopf algebras, including quantum groups
- Foliations, groupoids
- Singular spaces
- Deformation and quantization theory
- Topics in physics: quantum field theory, including noncommutative field theory and renormalization; gauge theories, including the Standard Model; condensed matter; gravity; strings

## 5.3 Spectral triples

The root of the importance of spectral triples in NCG is found in *algebraic topology*. Noncommutative topology brings techniques of operator algebra to algebraic topology —and vice versa. As indicated earlier, the method of rephrasing concepts and results from topology using Gelfand–Naimark and Serre–Swan equivalence, and extending them to some category of noncommutative algebras, recurs for a while. Moreover, deeper proofs of some properties of objects in the commutative world are to be found in their noncommutative counterparts, with Bott periodicity providing an outstanding example.

Now, to extend the standard (co)homology functors (not to speak of homotopy) is rather difficult. On the other hand, Atiyah’s  $K$ -functor generalizes very smoothly. Given a unital algebra  $A$ , its algebraic  $K_0$ -group is defined as the Grothendieck group of the (direct sum) semigroup of isomorphism classes of finitely generated projective right (or left) modules over  $A$ . Then in view of the Serre–Swan theorem  $K_0(C(X)) = K^0(X)$ .

Given an ordinary space  $X$ , the real  $K$ -group  $KO^0(X)$  —actually, it is a ring, with product given by pullback by the diagonal map of the tensor product— for  $X$  is obtained as the Grothendieck group for real vector bundles. Higher order groups are defined by suspension. If  $X$  is Hausdorff and compact, we have  $KO^i(X) \simeq KO^{i+8}(X)$ ; this is real Bott periodicity. Recall that we have:  $KO^0(*) = \mathbb{Z}, KO^1(*) = KO^2(*) = \mathbb{Z}_2, KO^3(*) = 0, KO^4(*) = \mathbb{Z}, KO^5(*) = KO^6(*) = KO^7(*) = 0$ . There is an isomorphism of the spin cobordism classes of a manifold  $X$  onto  $KO^\bullet(X)$  [62].

The  $K$ -homology of topological spaces can be developed as a functorial theory whose cycles pair with vector bundles in the same way that currents pair with differential forms in the de Rham theory. Such cycles are given, interestingly enough, by  $\text{spin}^c$  structures. On the other hand, the index theorem shows that the right partners for vector bundles are elliptic pseudodifferential operators (with the pairing given by the index map). We can think of abstract  $K$ -cycles as of phases of Dirac operators. In NCG we want to generalize both this and the line element (entering the realm of Riemannian geometry). Note the result:

**Proposition 1.** *On a spin manifold the geodesic distance between two points obeys the formula*

$$d(p, q) = \sup\{|f(x) - f(y)| : f \in C(X), |[D, f]| \leq 1\}. \quad (36)$$

This is actually trivial, since  $|[D, f]|$  is the Lipschitz norm of  $f$ .

The foregoing motivates:

**Definition 1.** A noncommutative geometry (spectral triple) is a triple  $(\mathcal{A}, H, D)$ , where  $\mathcal{A}$  is a  $*$ -algebra represented faithfully by bounded operators on the Hilbert space  $H$  and  $D$  is a self-adjoint operator  $D : \text{Dom} D \rightarrow H$ , with  $\text{Dom} D = H$ , such that  $[D, a]$  extends to a bounded operator and  $a(1 + D^2)^{-1/2}$  is a compact operator, for any  $a \in \mathcal{A}$ ; plus a postulate set of conditions given below.

We do not explicitly indicate the representation in the notation. A spectral triple is *even* when there exists on  $H$  a symmetry  $\Gamma$  such that  $\mathcal{A}$  is even and  $D$  odd with respect to the associated grading. Otherwise, it is odd. A spectral triple is *compact* when  $\mathcal{A}$  is unital; it is then enough to require that  $(1 + D^2)^{-1/2}$  be compact.

One should think of  $\mathcal{A}$  as of an algebra of ‘smooth’, not ‘continuous’ elements. Of course, it is important that  $K(\mathcal{A}) = K(\mathcal{A}^\sim)$ , with  $\mathcal{A}^\sim$  the  $C^*$ -algebra completion of  $\mathcal{A}$ . Sufficient conditions are known for this.

In the compact case the maximal set of postulates includes:

1. *Summability or Dimension:* for a fixed positive integer  $p$ , we have

$$(1 + D^2)^{-1/2} \in L^{p,+}(H), \quad \text{implying} \quad \text{Tr}_\omega((1 + D^2)^{-p/2}) \geq 0,$$

for all generalized limits  $\omega$ ; and moreover  $\text{Tr}_\omega((1 + D^2)^{-p/2}) \neq 0$ .

If we have regularity (see directly below), then the functional on  $\mathcal{A}$ :

$$a \mapsto \text{Tr}_\omega(a(1 + D^2)^{-p/2})$$

is a hypertrace.

2. *Regularity:* with  $\delta a := [D, a]$ , one has

$$\mathcal{A} \cup [D, \mathcal{A}] \subseteq \bigcap_{m=1}^{\infty} \text{Dom} \delta^m.$$

3. *Finiteness*: the dense subspace of  $H$  which is the smooth domain of  $D$ ,

$$H_\infty := \bigcap_{m \geq 1} \text{Dom } D^m$$

is a finitely generated projective (left)  $\mathcal{A}$ -module, which carries an  $\mathcal{A}$ -valued Hermitian pairing  $(\cdot | \cdot)_{\mathcal{A}}$  satisfying

$$\langle \xi | a\eta \rangle = \text{Tr}_\omega(a(\xi | \eta)_{\mathcal{A}}(1 + D^2)^{-p/2})$$

when  $\xi, \eta \in H_\infty$  and  $a \in \mathcal{A}$ . This also implies the *absolute continuity* property of the hypertrace:

$$\text{Tr}_\omega(a(1 + D^2)^{-p/2}) > 0, \quad \text{whenever } a > 0 \text{ in } \mathcal{A}.$$

4. *First-order condition*: as well as the defining representation we require a commuting representation of the opposite algebra  $\mathcal{A}^\circ$ . Now  $H_\infty$  can be regarded as a right  $\mathcal{A}$ -module. Then we furthermore ask for  $[[D, a], b] = 0$  for  $a \in \mathcal{A}, b \in \mathcal{A}^\circ$ . (When  $\mathcal{A}$  is commutative, we could still have different left and right actions on  $H$ . If they are equal, the postulate entails that the subalgebra  $\mathcal{C}_D \mathcal{A}$  of  $\mathcal{B}(H)$  generated by  $\mathcal{A}$  and  $[D, \mathcal{A}]$  belongs in  $\text{End}_{\mathcal{A}}(H_\infty)$ .)
5. *Orientation*: let  $p$  be the metric dimension of  $(\mathcal{A}, H, D)$ . We require that the spectral triple be even if and only if  $p$  is even. For convenience, we take  $\Gamma = 1$  when  $p$  is odd. We say the spectral triple  $(\mathcal{A}, H, D)$  is *orientable* if there exists a Hochschild  $p$ -cycle

$$\mathbf{c} = \sum_{\alpha=1}^n (a_\alpha^0 \otimes b_\alpha) \otimes a_\alpha^1 \otimes \cdots \otimes a_\alpha^p \in Z_p(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$$

whose Hochschild class may be called the “orientation” of  $(\mathcal{A}, H, D)$ , such that

$$\pi_D(\mathbf{c}) := \sum_{\alpha} a_\alpha^0 b_\alpha [D, a_\alpha^1] \cdots [D, a_\alpha^p] = \Gamma. \quad (37)$$

6. *Reality*: there is an antiunitary operator  $C : \mathcal{H} \rightarrow \mathcal{H}$  such that  $Ca^*C^{-1} = a$  for all  $a \in \mathcal{A}$ ; and moreover,  $C^2 = \pm 1$ ,  $CDC^{-1} = \pm D$  and also  $C\Gamma C^{-1} = \pm \Gamma$  in the even case, according to the following table of signs depending only on  $p \bmod 8$ :

$p \bmod 8$	0	2	4	6
$C^2 = \pm 1$	+	-	-	+
$CDC^{-1} = \pm D$	+	+	+	+
$C\Gamma C^{-1} = \pm \Gamma$	+	-	+	-

$p \bmod 8$	1	3	5	7
$C^2 = \pm 1$	+	-	-	+
$CDC^{-1} = \pm D$	-	+	-	+

For the origin of this sign table in  $KR$ -homology, we refer to [63]. (This postulate is optional, but important in practice. It makes the difference between  $\text{spin}^c$  and spin manifolds.)

7. *Poincaré duality*: the  $C^*$ -module completion of  $H_\infty$  is a Morita equivalence bimodule between  $\mathcal{A}$  and the norm completion of  $\mathcal{C}_D\mathcal{A}$ .

With the exception of the last, they are essentially in the form given to them by Connes.

What good are these terms? We have the following:

**Proposition 2.** *Let  $M$  be a compact Riemannian manifold without boundary with Riemannian volume form  $v_g$ , and assume there exists a spinor bundle  $S$  over it, with conjugation  $C$ . Define the **Dirac spectral triple** associated with it as*

$$(C^\infty(M), L^2(M, S), \not{D}),$$

where  $L^2(M, S)$  is the spinor space obtained by completing the spinor module  $\Gamma^\infty(M, S)$  with respect to the natural scalar product (using  $|v_g|$ ) and  $\not{D} := -i(\hat{e} \circ \nabla^S)$  is the Dirac operator (for the notation: if  $c$  is the action of the Clifford algebra bundle over  $M$ , then  $\hat{e}(\alpha, s) = c(\alpha)s$ , for  $\alpha$  in that bundle and  $s$  a spinor). Also  $\Gamma = c(\gamma)$ , where  $\gamma$  is the chirality element of the Clifford bundle, either the identity operator or the standard grading operator on  $L^2(M, S)$ , according as  $\dim M$  is odd or even.

Then the Dirac spectral triple is a commutative noncommutative spin geometry. (Sorry for the bad joke!)

The proof is routine. We can relax postulate 6 and obtain just a  $\text{spin}^c$  geometry. The most important thing is to think of the spinor bundle as an algebraic object: this comes from Plymen's characterization [64], suggested by Connes, of  $\text{spin}^c$  structures as Morita equivalence bimodules for the Clifford action induced by the metric. The existence of that equivalence is tantamount to the vanishing of the usual topological obstruction to the existence of  $\text{spin}^c$  structures. A precedent for this algebraization is Karrer's [65]. A recent article by Trautman [66] contains interesting historical asides.

## 5.4 On the reconstruction theorem

So far, so good, but there will be a point to the precedent exercise only if we can prove that the algebraic terms of the previous section lead in an essentially unique way to a spin manifold. That is, assuming conditions 1 to 7, excluding 6 for the time being, and furthermore that  $\mathcal{A}$  is commutative (this of course entails some simplification in the orientation axiom), is there a  $\text{spin}^c$  manifold  $M$ —with  $\dim M = p$ —such that  $A \simeq C^\infty(M)$  and similarly all of the original spectral triple is reproduced by its Dirac geometry?

Proof of this on the assumption that  $A \simeq C^\infty(M)$  for some  $M$  is found already in [63]. An attempt to prove it without that strong assumption was announced in

October 2006 by A. Rennie and J. C. Várilly [67]. However, this work had some flaws, recently corrected by Connes [68, 69].

Some extra technical assumptions are needed for the proof. Rennie and Várilly assume that the spectral triple  $(\mathcal{A}, H, D)$  is *irreducible*, that is, the only operators in  $\mathcal{B}(\mathcal{H})$  commuting (strongly) with  $D$  and with all  $a \in \mathcal{A}$  are the scalars in  $\mathbb{C}1$ . (This ensures the connectedness of the underlying topological space  $M$ .) Moreover, they postulate the following *closedness* condition: for any  $p$ -tuple of elements  $(a_1, \dots, a_p)$  in  $\mathcal{A}$ , the operator  $\Gamma[D, a_1] \dots [D, a_p](1 + D^2)^{-p/2}$  has vanishing Dixmier trace; thus, for any  $\omega$ ,

$$\mathrm{Tr}_\omega(\Gamma[D, a_1] \dots [D, a_p](1 + D^2)^{-p/2}) = 0.$$

This is an algebraic analogue of Stokes’ theorem.

Their argument to show that the Gelfand–Naïmark spectrum  $M$  of  $\mathcal{A}$  is a differential manifold may be conceptually broken into two stages. The first is to construct a vector bundle over the spectrum which will play the role of the cotangent bundle. For that, one identifies local trivializations and bases of this bundle in terms of the ‘1-forms’  $[D, a_\alpha^j]$  given by the orientability condition. The aim is then to show that the maps  $a_\alpha = (a_\alpha^1, \dots, a_\alpha^p) : M \rightarrow \mathbb{R}^p$  provide coordinates on suitable open subsets of  $M$ ; for that, one must prove that the maps  $a_\alpha$  are open and locally one-to-one.

At this stage one needs to deploy, besides the technical conditions, postulates 1 to 5 on our spectral triple. A basic tool is a multivariate  $\mathbb{C}^\infty$  functional calculus for regular spectral triples, that enables to construct partitions of unity and local inverses within the algebra  $\mathcal{A}$ .

However, the strategy of [67] failed to ensure that the maps  $a_\alpha$  are local homeomorphisms. Instead, Connes [69] resorted to the inverse function theorem [70], by showing that regularity and finiteness provide enough smooth derivations of  $\mathcal{A}$  to build nonvanishing Jacobians where needed. This requires delicate arguments with unbounded derivations of  $C^*$ -algebras, and two other technical assumptions, replacing those of [67]:

- *Skewsymmetry* of the Hochschild cycle  $c$  under permutations of  $a_\alpha^1, \dots, a_\alpha^p$ . This enables one to bypass the cotangent bundle construction and omit the closedness property, but is arguably a stronger assumption.
- *Strong regularity*: all elements of  $\mathrm{End}_\mathcal{A}(H_\infty)$ , not merely those in  $\mathcal{C}_D\mathcal{A}$ , lie in  $\bigcap_{m=1}^\infty \mathrm{Dom} \delta^m$ .

The local injectivity of the maps  $a_\alpha$  is established by first showing that their multiplicity (as maps into  $\mathbb{R}^p$ ) is bounded: this needs delicate estimates in order to invoke the measure theoretic results of Voiculescu [71]. The smooth functional calculus can then be used to construct local charts at all points of  $M$  by small shifts of the original maps  $a_\alpha$ .

Poincaré duality in  $K$ -theory plays no role in the reconstruction of a manifold as a compact space  $M$  with charts and smooth transition functions. However, once that has been achieved, it is needed to show that  $M$  carries a  $\mathrm{spin}^c$  structure and to identify the class of  $(\mathcal{A}, H, D)$  as the fundamental class of the  $\mathrm{spin}^c$  manifold. This is

done by showing that in this case  $\text{End}_{\mathcal{A}}(H_\infty)$  coincides with  $\mathcal{C}_D \mathcal{A}$ —see [67, 69]—and in particular strong regularity is moot. The Dirac operator is shown to differ from  $D$  by at most an endomorphism of the corresponding spinor bundle. When  $M$  is spin, the latter can be eliminated by a variational argument—as shown by Kastler, and by Kalau and Walze, the Wodzicki residue of  $(1 + D^2)^{-p/2+1}$  gives the EH action; see [63, Sect. 11.4].

Once one has at one's disposal a  $\text{spin}^c$  structure, axiom 6 (Reality) allows to refine it to a spin structure. For that, we refer to [64]—or consult [63]—wherein it is shown that the spinor module for a spin structure is just the spinor module for a  $\text{spin}^c$  structure equipped with compatible charge conjugation, which is none other than the real structure operator  $C$  (acting on  $H_\infty$ ); the spin structure is extracted, using  $C$ , from a representation of the real Clifford algebra of  $T^*M$ .

It is unlikely [72] that the reconstruction theorem holds under the more stringent conditions set out originally by Connes [73]. Possible redundancy of the system of postulates has not been much investigated; but certainly there are indications that the ones related with dimension are independent.

### 5.5 The noncommutative torus

This was the early paradigm for nc manifolds, where everything works smoothly. For a fixed irrational real number  $\theta$ , let  $A_\theta$  be the unital  $C^*$ -algebra generated by two elements  $u, v$  subject only to the relations  $uu^* = u^*u = 1$ ,  $vv^* = v^*v = 1$ , and

$$vu = \lambda uv \quad \text{where} \quad \lambda := e^{2\pi i \theta}. \quad (38)$$

Let  $\mathcal{S}(\mathbb{Z}^2)$  denote the double sequences  $\underline{a} = \{a_{rs}\}$  that are *rapidly decreasing* in the sense that

$$\sup_{r,s \in \mathbb{Z}} (1 + r^2 + s^2)^k |a_{rs}|^2 < \infty \quad \text{for all} \quad k \in \mathbb{N}.$$

The irrational rotation algebra or *noncommutative torus algebra*  $T_\theta$  is defined as

$$T_\theta := \left\{ a = \sum_{r,s} a_{rs} u^r v^s : \underline{a} \in \mathcal{S}(\mathbb{Z}^2) \right\}.$$

It is a *pre- $C^*$ -algebra* that is dense in  $A_\theta$ . The product and involution in  $T_\theta$  are computable from (38):

$$ab = \sum_{r,s} a_{r-n,m} \lambda^{mn} b_{n,s-m} u^r v^s, \quad a^* = \sum_{r,s} \lambda^{rs} \bar{a}_{-r,-s} u^r v^s.$$

The irrational rotation algebra gets its name from another representation, on  $L^2(T)$ : the multiplication operator  $U$  and the rotation operator  $V$  given by  $(U\psi)(z) := z\psi(z)$  and  $(V\psi)(z) := \psi(\lambda z)$  satisfy (38). In the  $C^*$ -algebraic framework,  $U$  generates the  $C^*$ -algebra  $C(T)$  and conjugation by  $V$  gives an automorphism  $\alpha$  of  $C(T)$ . Under

such circumstances, the  $C^*$ -algebra generated by  $C(T)$  and the unitary operator  $V$  is called the *crossed product* of  $C(T)$  by the automorphism group  $\{\alpha^n : n \in \mathbb{Z}\}$ . In symbols,

$$A_\theta \simeq C(T) \times_\alpha \mathbb{Z}.$$

The corresponding action by the rotation angle  $2\pi\theta$  on the circle is ergodic and minimal (all orbits are dense); it is known that the  $C^*$ -algebra  $A_\theta$  is therefore simple.

Using the abstract presentation by (38), certain *isomorphisms* become evident. First of all,  $T_\theta \simeq T_{\theta+n}$  for any  $n \in \mathbb{Z}$ , since  $\lambda$  is the same for both. Next,  $T_\theta \simeq T_{-\theta}$  via the isomorphism determined by  $u \mapsto v$ ,  $v \mapsto u$ . There are no more isomorphisms among the  $T_\theta$ .

The linear functional  $\tau_0 : T_\theta \rightarrow \mathbb{C}$  given by  $\tau_0(a) := a_{00}$  is positive definite since  $\tau_0(a^*a) = \sum_{r,s} |a_{rs}|^2 > 0$  for  $a \neq 0$ ; it satisfies  $\tau_0(1) = 1$  and is a trace, since  $\tau_0(ab) = \tau_0(ba)$ . Also, it can be shown that  $\tau_0$  extends to a faithful continuous trace on the  $C^*$ -algebra  $A_\theta$ ; and, in fact, this normalized trace on  $A_\theta$  is unique. The GNS representation space  $\mathcal{H}_0 = L^2(T_\theta, \tau_0)$  may be described as the completion of the vector space  $T_\theta$  in the Hilbert norm  $\|a\|_2 := \sqrt{\tau_0(a^*a)}$ . Since  $\tau_0$  is faithful, the obvious map  $T_\theta \rightarrow \mathcal{H}_0$  is injective; to keep the bookkeeping straight, in this section we shall denote by  $\underline{a}$  the image in  $\mathcal{H}_0$  of  $a \in T_\theta$ . The GNS representation of  $T_\theta$  is just  $\underline{b} \mapsto \underline{ab}$ . The vector  $\underline{1}$  is obviously cyclic and separating, and the Tomita involution is given by  $J(\underline{a}) := \underline{a^*}$ , thus  $J = J^\dagger$ . The commuting representation is then given by

$$b \mapsto J\pi(a^*)J^\dagger \underline{b} = J\underline{a^*b^*} = \underline{ba}.$$

To build a two-dimensional geometry, we need to have a  $\mathbb{Z}_2$ -graded Hilbert space on which there is an antilinear involution  $C$  that anticommutes with the grading and satisfies  $C^2 = -1$ . There is a simple device that solves all of these requirements: we simply double the GNS Hilbert space by taking  $H := H_0 \oplus H_0$  and define

$$C := \begin{pmatrix} 0 & -J \\ J & 0 \end{pmatrix}.$$

In order to have a spectral triple, it remains to introduce the operator  $D$ . For  $D$  to be selfadjoint and anticommute with  $\Gamma$ , it must be of the form

$$D = -i \begin{pmatrix} 0 & \underline{\partial}^\dagger \\ \underline{\partial} & 0 \end{pmatrix},$$

for a suitable closed operator  $\underline{\partial}$  on  $L^2(T_\theta, \tau_0)$ . The order-one axiom, together with the regularity axiom and the finiteness property lead to  $\underline{\partial}, \underline{\partial}^\dagger$  being derivations of  $T_\theta$ . The reality condition  $CDC^\dagger = D$  is equivalent to the condition that  $J\underline{\partial}J = -\underline{\partial}^\dagger$  on  $L^2(T_\theta, \tau_0)$ . Consider the derivations

$$\delta_1(a_{rs}u^r v^s) := 2\pi i r a_{rs} u^r v^s; \quad \delta_2(a_{rs}u^r v^s) := 2\pi i s a_{rs} u^r v^s.$$

For concreteness, take  $\underline{\partial}$  to be a linear combination of the basic derivations basic derivations  $\delta_1, \delta_2$ . Apart from a scale factor, the most general such derivation is

$\partial = \partial_\tau := \delta_1 + \tau \delta_2$  with  $\tau \in \mathbb{C}$ . In fact, real values of  $\tau$  must be excluded. Now,  $D_\tau^{-2}$  has discrete spectrum of eigenvalues  $(4\pi^2)^{-1}|m+n\tau|^{-2}$ , each with multiplicity 2. The Eisenstein series  $\sum_{m,n \neq 0,0} \frac{1}{(m+n\tau)^2}$  diverges logarithmically, thereby establishing the two-dimensionality of the geometry. The orientation cycle is given by

$$\frac{1}{4\pi^2(\tau - \bar{\tau})} (v^{-1}u^{-1} \otimes u \otimes v - u^{-1}v^{-1} \otimes v \otimes u).$$

This makes sense only if  $\tau - \bar{\tau} \neq 0$ , i.e.,  $\tau \notin \mathbb{R}$ . Thus  $(\Im \tau)^{-1}$  is a scale factor in the metric determined by  $D_\tau$ . (Note a difference with the commutative volume form: since  $v^{-1}u^{-1} = \lambda u^{-1}v^{-1}$ , there is also a phase factor  $\lambda = e^{2\pi i \theta}$  in the orientation cycle.)

We conclude by indicating that the noncommutative torus can be regarded as well as a deformation, as it corresponds to the Moyal product of periodic functions. There are of course nc tori of all dimensions greater than 2.

## 5.6 The noncompact case

Real *noncompact spectral triples* (also called nonunital spectral triples) have implicitly been already defined. In practice the data are of the form

$$(\mathcal{A}, \widetilde{\mathcal{A}}, H, D; C, \Gamma),$$

where now  $\mathcal{A}$  is a nonunital algebra and the new element  $\widetilde{\mathcal{A}}$  is a preferred unitization of  $\mathcal{A}$ , acting on the same Hilbert space.

To get an idea of the difficulties involved in the choice of  $\mathcal{A}$ , consider the simplest commutative case, say of the manifold  $\mathbb{R}^p$ . Depending on the fall-off conditions deemed suitable, the smooth nonunital algebras that can represent the manifold are numerous as the stars in the sky. The problem is compounded in the noncommutative case, say when  $\mathcal{A}$  is a deformation of an algebra of functions. To be on the safe side, one should take a relatively small algebra at the start of any investigation of examples.

Postulates 2, 4 and 6 need no changes with respect to the compact case formulation.

Now, we ponder:

- Dimension of the geometry: for  $p$  a positive integer  $a(1 + D^2)^{-1/2}$  belongs to the generalized Schatten class  $\mathcal{L}^{p,+}$  for each  $a \in \mathcal{A}$ , and moreover  $\text{Tr}_\omega(a(1 + |D|)^{-p})$  is finite and not identically zero.
- Finiteness: the algebra  $\mathcal{A}$  and its preferred unitization  $\widetilde{\mathcal{A}}$  are pre- $C^*$ -algebras. There exists an ideal  $\mathcal{A}_1$  of  $\widetilde{\mathcal{A}}$ , including  $\mathcal{A}$ , which is also a pre- $C^*$ -algebra with the same  $C^*$ -completion as  $\mathcal{A}$ , such that the space of smooth vectors is an  $\mathcal{A}_1$ -pullback of a finitely generated projective  $\widetilde{\mathcal{A}}$ -module. Moreover, an  $\mathcal{A}_1$ -valued



hermitian structure is defined on  $H_\infty$  with the noncommutative integral; this is an absolute continuity condition.

- Orientation: there is a *Hochschild  $p$ -cycle*  $\mathbf{c}$  on  $\widetilde{\mathcal{A}}$ , with values in  $\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^\circ$ . Such a  $p$ -cycle is a finite sum of terms like  $(a^0 \otimes b) \otimes a^1 \otimes \cdots \otimes a^p$ , whose natural representative by operators on  $\mathcal{H}$  is given by  $\pi_D(\mathbf{c})$  in formula (37); the volume form  $\pi_D(\mathbf{c})$  must solve the equation

$$\pi_D(\mathbf{c}) = \Gamma \quad (\text{even case}), \quad \text{or} \quad \pi_D(\mathbf{c}) = 1 \quad (\text{odd case}).$$

The need for some preferred unitization is plain, as finiteness requires the presence both of a nonunital and a unital algebra. Then examples show the need for a further subtlety, to wit, the nonunital algebra for which summability works is *smaller* than the nonunital algebra required for finiteness. Also, orientation is defined directly on the preferred unitization.

The commutative examples were worked out in [74, 75]; there summability works in view of asymptotic spectral analysis for the Dirac operator. In [76] —to some surprise of Alain Connes— it was shown that Moyal algebras are noncompact spectral triples.

It is worthwhile to point out that the NCG versions of the Standard Model are noncompact spectral triples, too; while there is no end of algebraic intricacies for the finite dimensional representation [77] required to reproduce the quirks of particle physics, analytically the problem is to be tackled by the methods of the mentioned papers [74, 75, 76].

### 5.7 *Nc toric manifolds (compact and noncompact)*

How does one recover the metric geometry of the Riemann sphere  $\mathbb{S}^2$  from spectral triple data? If  $\mathcal{A}$  is a dense subalgebra of a some  $C^*$ -algebra containing elements  $x, y, z$  and if the matrix

$$p = \frac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}$$

is a projector, it is easy to see from the projector relations that  $x, y, z$  commute and that  $x^2 + y^2 + z^2 = 1$ . Thus  $A = C(X)$  where  $X \subset \mathbb{S}^2$  is closed. The condition

$$\pi_D \left( \text{tr} \left( (p - \tfrac{1}{2}) \otimes p \otimes p \right) \right) = \Gamma$$

can only hold if  $X = \mathbb{S}^2$ . In the same way, Connes sought to obtain the sphere  $\mathbb{S}^4$  with its round metric by starting with an analogous projector in  $M_4(\mathcal{A})$ :

$$p = \begin{pmatrix} (1+z)1_2 & q \\ q & (1-z)1_2 \end{pmatrix},$$

with  $q$  the quaternion

$$q = \begin{pmatrix} a & b \\ -b^* & a \end{pmatrix},$$

imposing conditions so that

$$\pi_D \left( \text{tr} \left( (p - \tfrac{1}{2}) \otimes p \otimes p \otimes p \otimes p \right) \right) = \Gamma.$$

Again  $\mathcal{A}$  is commutative and the 4-sphere relation holds. But then Landi surprised everyone by pointing out that one could substitute  $-\lambda b^*$  for the entry  $-b^*$ . With  $\lambda = e^{2\pi i \theta}$ , this works into a spectral triple. It was called an *isospectral deformation* because the Dirac operator remains untouched [78].

Again, this generalizes into a  $\theta$ -deformation of any Riemannian manifold  $M$  that admits  $T^2$  as a subgroup of its group of isometries. And again, this is essentially a Moyal deformation: if  $M = G/K$ , with  $G$  compact of rank at least two, then  $C^\infty(G)$  can be deformed in such a way that  $C^\infty(M_\theta)$  is a homogenous space of the compact quantum group  $C^\infty(G_\theta)$  [79].

The procedure can be generalized to a large family of noncompact Riemannian spin manifolds (with ‘bounded geometry’) that admit an action of  $T^l$ , for  $l \geq 2$ , or a free action of  $\mathbb{R}^l$ , for  $l \geq 2$  [80]. The upshot is more noncommutative spin geometries.

(Even lowly  $\mathbb{S}^2$  hides surprises, too, if one allows for relaxing the notion of what a Dirac operator is [81].)

## 5.8 Closing points

### 5.8.1 Fabricating nc spaces: a second conceptual star and catalogue

So far, we have played it very safe, and we have said little on how to handle wilder examples of nc manifolds. Connes himself recommends the following steps [82]:

1. Given an algebra  $\mathcal{A}$  (putative ‘of smooth functions on a nc manifold’), try first of finding a resolution of it as an  $\mathcal{A}$ -bimodule, with a view to compute its Hochschild cohomology, and eventually its cyclic homology and cohomology. This is not an easy task in general; it has been performed in the commutative case and for foliations.
2. Many nc spaces arise as ‘bad quotients’. Consider  $Y := X / \sim$ . If one tries to study

$$C(Y) = \{ f \in C(X) : f(a) = f(b), \forall a \sim b \},$$

one often ends up with only constant functions. (It is true that, for proper actions of Lie groups, even if  $M/G$  is not a manifold, there is however an interesting functional structure [83, 84], that can be usefully studied by a mixture of “commutative” and “noncommutative” methods.) It beckons to drop the commutativity requirement by considering complex functions of two variables defined on

the graph of the equivalence relation. They will act as bounded operators on the Hilbert space of the equivalence class, and they multiply with the convolution product:

$$(fg)_{ab} = \sum_{a \sim c \sim b} f_{ac} g_{cb}. \quad (39)$$

Of course, when the quotient space is ‘nice’, one can do that, too; as a rule in this case, the commutative and noncommutative algebras are Morita equivalent. But in a case as simple as  $X = [0, 1] \times \mathbb{Z}_2$  with  $\sim$  given by  $(x, +) \sim (x, -)$  for  $x \in ]0, 1[$ , we obtain for the convolution algebra the “dumbbell” algebra:

$$\{f \in C([0, 1]) \otimes M_2\mathbb{C} : f(0), f(1) \text{ diagonal}\};$$

and there is no such equivalence. The idea is then to compute the  $K$ -theory, in order to learn as much as possible on the space. Ideally, one would also like to have ‘vector bundles’, Chern character (using connections and curvature) and even moduli spaces for Yang-Mills connections —this works wonderfully for nc tori, which after all are quotient spaces.

Incidentally, families of maps that are semigroups in the commutative word naturally become  $C^*$ -bialgebras in the noncommutative context. We may refer to the recent beautiful paper by Soltan [85], where the quantum family of maps from  $\mathbb{C}^2$  to  $\mathbb{C}^2$  is identified to the dumbbell algebra.

Let us add as well that Connes contends that the foundational step of Quantum Mechanics (by Heisenberg in 1925) amounts to replacing an abelian group law by a groupoid law like (39), in order to make sense of the combination principles of spectral lines.

3. Then come the spectral triples. They respond for  $K$ -homology classes, smooth structure, and metric. There is a surprisingly vast class of spaces that can be described in this way, under conditions in general less strict than the ones required for the reconstruction theorem.
4. The time evolution and thermodynamic aspects.

That said, we can prepare our catalogue (leaving aside subjects related to physics, for a moment):

- Spaces of leaves of foliations. This was an early, successful application of nc geometry. By elaborating on the construction of point 2 above, Connes was able to apply methods of operator theory to foliation theory.
- Tilings (periodic and aperiodic). Also under point 2.
- Dynamical systems. Also point 2.
- Cantor sets and fractals. One can associate spectral triples (Dirac operators) to them! The algebra of continuous functions on a Cantor set is AF commutative. We omit the details on the construction of  $(H, D)$ . It is then very interesting to investigate the dimension spectrum of the spectral triple. For the classical middle-third Cantor set:

$$\mathrm{Tr}(|D|^{-s}) = 2 \sum_k l_k^s = \sum_{k \geq 1} 2^k 3^{-sk} = \frac{23^{-s}}{1 - 23^{-s}},$$

given that  $l_k = 3^{-k}$  with multiplicities  $2^{k-1}$ . This yields as dimension spectrum

$$\frac{\log 2}{\log 3} + \frac{2\pi i n}{\log 3},$$

for  $n \in \mathbb{Z}$ . For compact fractal subsets of  $\mathbb{R}^n$ . Christensen and Ivan recently have constructed spectral triples not satisfying Weyl's asymptotic formula —there is no constant  $c$  so that the number of eigenvalues  $N(\Lambda)$  bounded by  $\Lambda$  fulfils

$$N(\Lambda) - c\Lambda \sim \text{lower order in } \Lambda.$$

- Algebraic deformations. Of this the Moyal-like spaces are the outstanding example. More on that below.
- Spherical manifolds which are not isospectral deformations. I refer to [86] and subsequent papers by Connes and Dubois-Violette.
- Nc spaces related to arithmetic problems (including some that have been used by Connes to try to prove the Riemann hypothesis). On this I claim zero expertise.

### 5.8.2 What about physics?

- Quantum Hall effect, related to nc tori. This is due to Bellissard.
- Nc spaces from axiomatic QFT. For instance, the local algebras in a supersymmetric model, together with the supercharge as a Dirac operator, constitute a spectral triple.
- Nc spaces from renormalization, via dimensional regularization. This has been only hinted at.
- The mentioned Standard Model reconstruction from NCG.
- Nc spaces from strings. If one goes to the physics archives and asks for “noncommutative geometry” or “noncommutative field theory”, what one finds is something as puzzling as particular, that is, perturbative quantum field theory over Moyal hyperplanes. This was popularized by Seiberg and Witten [87] as a certain limit of string theory, but has acquired a life of its own. Nevertheless [76] and subsequent papers [88, 89] tried to make a bridge between this and Connes' paradigm.

### 5.8.3 Some neglected tools

- Lie algebroids, Lie–Rinehart algebras and the like. It is a little mystery why, while groupoids play a central role in NCG, their infinitesimal version does not seem to play any role. All the more so because the algebraic version of Lie algebroids, the theory of Lie–Rinehart(–Gerstenhaber) algebras, which seems to be the good framework for BRS theory, has very much the flavour of NCG, and is quite able to deal with many singular spaces [90].

Lie–Rinehart algebras are usually commutative; but some of the results pertaining to them can be extended to “softly noncommutative” cases. Most importantly, the theory of Adams operations, that plays such an important role in the Hochschild and cyclic cohomology of commutative algebras, can be extended to the realm of noncommutative spaces [91]. This connects the local index formula by Connes and Moscovici [92] with combinatorial aspects (the Dynkin operator of free Lie algebra theory and noncommutative symmetric functions) that have not been fully explored.

- Rota–Baxter operators and skewderivations. A poor man’s path to the nc world (akin to the one taken by some quantum group theorists) is to try to generalize the usual derivative/integral pair. This is elementary stuff with many ramifications. A skewderivation of weight  $\theta \in \mathbb{R}$  is a linear map  $\delta : A \rightarrow A$  fulfilling the condition

$$\delta(ab) = a\delta(b) + \delta(a)b - \theta\delta(a)\delta(b). \quad (40)$$

We may call skewdifferential algebra a double  $(A, \delta; \theta)$  consisting of an algebra  $A$  and a skewderivation  $\delta$  of weight  $\theta$ . A *Rota–Baxter map*  $R$  of weight  $\theta \in \mathbb{R}$  on a not necessarily associative algebra  $A$ , commutative or not, is a linear map  $R : A \rightarrow A$  fulfilling the condition

$$R(a)R(b) = R(R(a)b) + R(aR(b)) - \theta R(ab), \quad a, b \in A. \quad (41)$$

When  $\theta = 0$  we obtain the integration-by-parts rule. The triple  $(A, \delta, R; \theta)$  will denote an algebra  $A$  endowed with a skewderivation  $\delta$  and a corresponding Rota–Baxter map  $R$ , both of weight  $\theta$ , such that  $R\delta a = a$  for any  $a \in A$  such that  $\delta a \neq 0$ , as well as  $\delta Ra = a$  for any  $a \in A, Ra \neq 0$ . We can check consistency of conditions (40) and (41) imposed on  $\delta, R$ :

$$\begin{aligned} \theta\delta R(ab) &= R(a)b + aR(b) - \delta(R(a)R(b)) \\ &= R(a)b + aR(b) - R(a)b - aR(b) + \theta ab = \theta ab; \\ R\delta(ab) &= R(a\delta(b)) + R(\delta(a)b) - \theta R(\delta(a)\delta(b)) = R(a\delta(b)) + R(\delta(a)b) \\ &\quad - R(a\delta(b)) - R(\delta(a)b) + ab = ab. \end{aligned}$$

Rota–Baxter operators have proved their worth in probability theory and combinatorics, and in the Connes–Kreimer approach to renormalization; but their range of applications is much wider.

- What is the natural noncommutative algebra structure than one should impose on ordinary, well behaved manifolds? The author has long contended that the answer, at least in the equivariant case, is: general Moyal theory. Given the naturalness of ordinary Moyal quantization on hyperplanes, the high number of nc spaces that turn out to be related to Moyal quantization, plus the usefulness of Moyal quantization in proofs (for instance of Bott periodicity in the algebraic context), it is surprising that few nc geometers seem interested in general Moyal theory.

But how to define the latter? It would run as follows. Let  $X$  be a phase space,  $\mu$  a Liouville measure on  $X$ , and  $H$  the Hilbert space associated to  $(X, \mu)$ . A Moyal or Stratonovich–Weyl quantizer for  $(X, \mu, H)$  is a mapping  $\Omega$  of  $X$  into the space of selfadjoint operators on  $H$ , such that  $\Omega(X)$  is weakly dense in  $B(H)$ , and verifying

$$\begin{aligned}\mathrm{Tr} \Omega(u) &= 1, \\ \mathrm{Tr} [\Omega(u) \Omega(v)] &= \delta(u - v),\end{aligned}$$

in the distributional sense. (Here  $\delta(u - v)$  denotes the reproducing kernel for the measure  $\mu$ .) Moyal quantizers, if they exist, are unique, and ownership of a Moyal quantizer solves in principle all quantization problems: *quantization* of a (sufficiently regular) function or “symbol”  $a$  on  $X$  is effected by

$$a \mapsto \int_X a(u) \Omega(u) d\mu(u) =: Q(a),$$

and *dequantization* of an operator  $A \in B(H)$  is achieved by

$$A \mapsto \mathrm{Tr} A \Omega(\cdot) =: W_A(\cdot).$$

Indeed, it follows that  $1_H \mapsto 1$  by dequantization, and also

$$\mathrm{Tr} Q(a) = \int_X a(u) d\mu(u).$$

Moreover, using the weak density of  $\Omega(X)$ , it is clear that:

$$W_{Q(a)}(u) = \mathrm{Tr} \left[ \left( \int_X a(v) \Omega(v) d\mu(v) \right) \Omega(u) \right] = a(u),$$

so  $Q$  and  $W$  are inverses. In particular,  $W_{Q(1)} = 1$  says that  $1 \mapsto 1_H$  by quantization, and this amounts to the reproducing property  $\int_X \Omega(u) d\mu(u) = 1_H$ . Finally, we also have

$$\mathrm{Tr} [Q(a) Q(b)] = \int_X a(u) b(u) d\mu(u).$$

This is the key property. Most interesting cases occur in an equivariant context ; that is to say, there is a (Lie) group  $G$  for which  $X$  is a symplectic homogeneous  $G$ -space, with  $\mu$  then being a  $G$ -invariant measure on  $X$ , and  $G$  acts by a projective unitary irreducible representation  $U$  on the Hilbert space  $H$ . A Moyal quantizer for the combo  $(X, \mu, H, G, U)$  is a map  $\Omega$  taking  $X$  to selfadjoint operators on  $\mathcal{H}$  that satisfies the previous defining equations and the equivariance property

$$U(g) \Omega(u) U(g)^{-1} = \Omega(g \cdot u), \quad \text{for all } g \in G, u \in X.$$

The question is: how to find the quantizers? The fact that the solution in flat spaces leads to (bounded) parity operators points out to the framework of *symmetric spaces* as the natural one to find Moyal quantizers by interpolation. This heuristic parity rule was found to work for orbits of the Poincaré group [93]. Noncompact symmetric spaces should provide a wealth of noncompact spectral triples (the compact case is somewhat pathological). Recently the author, together with V. Gayral and J. C. Várilly, has given the Moyal quantization of the surface of constant negative curvature [94]; a new special function plays there the main role in framing a subtler version of the parity rule.

- Algebraic  $K$ -theory, noncommutative geometry and field theory. The role of the two first functors of algebraic  $K$ -theory in QFT with external fields is “well-known”; Connes has dabbled on this, but he has not pursued the subject. To this writer, also in relation with [92], it seems extremely promising.

## 5.9 Some interfaces with quantum gravity

This subsection is intended as a taunt. We just lift a corner of the veil.

### 5.9.1 Noncommutative field theory and quantum gravity

Direct connection between noncommutative field theory and quantum gravity has been sought in several papers. The basic idea is due to Rivelles [95]. In noncommutative *gauge* theories, translations are equivalent to gauge transformations. This at once reminds one of gravitation (the case can be made that translations necessarily involve gauge transformations in Yang–Mills theories, too [96]; but this is a weaker statement). In general, the distinction between internal and geometrical degrees of freedom fades in noncommutative geometry [97]. Indeed in [95] it is shown, using Seiberg–Witten maps [87], how the field action can be regarded as a coupling to a gravitational background. The idea has been further developed in [98]. In some other papers suggesting a noncommutative geometry formalism for pure classical gravity, the apparatus is so heavy as to make it difficult to see the forest for the trees [99]. A different approach is to look for noncommutative corrections to particular classes of spacetimes. This is found in [100]. The “barriers to entry” in this field being relatively modest, we cut our remarks short.

### 5.9.2 Isospectral deformations and unimodularity

There seems to be no good reason to exclude noncommutative manifolds in the sense of Connes from the approaches to quantum gravity based on “sum over geometries”. Already, in an important paper [101], Yang has showed that the Eguchi and Hanson gravitational instantons [102] give rise by isospectral deformation to

noncommutative noncompact manifolds in the sense of [76]. Now, isospectral deformation leaves the orientation condition unchanged. The general paradigm is as follows: *any* Dirac operator, describing a  $K$ -homology class, corresponding to a commutative manifold (thus, for any Riemannian geometry over it) or noncommutative one, solves equally well, and on the same footing, the “topological equation” that defines the manifold itself. With the proviso that the volume form remains the same. The advantages indicated in [57] should apply in this context, too.

The punch line: in its present form at least, noncommutative geometry favours the unimodular theory.

## 6 More on the “cosmological constant problem” and the astroparticle interface

Notice that both terminologies “cosmological constant” and “dark energy” betray theoretical prejudices.

The first name, that we can deal with the observations pointing to an acceleration of the expansion rate of the universe by just including the so-called cosmological term in the Einstein equations. In fact, we do not know the equation of state, not to speak of the evolution laws, of whatever exotic “substance” that might be involved [103].

The second is related to the belief that the acceleration be caused by fluctuations, or “zero-point energies” of the quantum vacuum, somehow. Alas, this notion here was entertained by nobody less than Weinberg, whose already mentioned [56] threw both light and obscurity on the subject.

The whole review hangs on the thread that there must be a problem, since:

... the energy density of the vacuum acts just like a cosmological constant.

However, the effective cosmological constant is quite small (we wouldn’t be here otherwise). On the face of it, zero-point energies are infinite (well, this is not the case in Epstein–Glaser renormalization, but let us go with the argument). If we take as a sensible cut-off the Planck scale, the amount of “fine-tuning” necessary to cancel their contribution is mind-boggling. Thus,

Perhaps surprisingly, it was a long time before particle physicists began seriously to worry about this problem, despite the demonstration in the Casimir effect of the reality of zero-point energies.

The trouble is, that “demonstration” is another urban legend. The negative weight of zero-point fluctuations is *unobserved* in any laboratory experiment, *including the Casimir effect*. The latter is measured nowadays well enough. However, the usual derivation in terms of differences of zero-point energies, and its neat result, where only  $c$ ,  $\hbar$  and the geometry of the plates enter, inviting us to think of it as a “property of the vacuum”, is misleading. The point has been made recently by Jaffe [104]. In truth, the Casimir effect distinguishes itself from other quantum electrodynamics



only in that (for some geometrical configurations, not for all) it reaches a finite limit as the fine structure constant  $\alpha \uparrow \infty$ ; this limit is the usually quoted result. In that derivation, the plates are treated as perfect conductors. However, a perfect conductor at all frequencies is a physical impossibility. The plasma frequency

$$\omega_{\text{pl}} = 2e\sqrt{\frac{\pi n}{m}}$$

indicates the frequency above which the conductivity goes to zero; here  $n$  is the density of conduction band electrons and  $m$  their effective mass. So the perfect conductor approximation is good if  $c/d \ll \omega_{\text{pl}}$ , with  $d$  the distance between plates; that is for materials and plate distances such that

$$\frac{1}{137} \sim \alpha \gg \frac{mc}{4\pi\hbar nd^2}.$$

Still, it remains an approximation. Casimir forces can be and have been calculated without reference to the vacuum. Whether there can be experimental evidence for zero point energies, apart from gravity, is an open question, which may be answered in the negative for all we know. The lesson is that their putative contribution to the cosmological constant must be in doubt. As Jaffe puts it [104]:

Caution is in order when an effect, for which there is no direct experimental evidence, is the source of a huge discrepancy between theory and experiment.

Indeed.

We might add: nowadays there is a “vacuum fluctuations” branch of mathematics, conductors which are always perfectly so and plates of vanishing thickness *etsi daretur*. This is to the good, and may be helpful, provided we keep the origins in mind and do not start to draw unwarranted physical inferences! We are reminded of Manin’s dicta. A mathematically rigorous and physically sound account of the Casimir effect without invoking “zero-point energies”, particularly unveiling the unphysical nature of Dirichlet boundary conditions, has been given by Herdegen [105].

Parenthetically, one finds in the work of Vachapasti and coworkers on “black stars” mentioned in the first section [7] a comendable retreat to consideration of *physical* black holes —collapsing bodies suspended above their Schwarzschild radius forever from a remote observer viewpoint— rather than mathematical black holes —vacuum solutions of the general relativity equations. While the mathematical study of black holes remains a useful and fascinating subject, the former is required to explain astrophysical observations.

On the other hand, it is hard to dispute that the energy density of the vacuum itself should act like a cosmological constant. Thus it is rather less clear why the flavourdynamics scale —whereby we are talking not of phantom fluctuations, but of the vacuum expected value of the energy itself— does not play a role. Even if one (as this writer) does not trust the Higgs mechanism, there is reason to worry about the contribution of chiral symmetry breaking in the quark condensate, still twelve

orders of magnitude above the “observed” range for the cosmological constant. For this reason unimodularity as discussed in Section 4 should be taken seriously.

A recommended review on the cosmological constant is [106]. Its author dismisses “fine-tuning” out of hand. Suggestive thinking on the dark energy problem is found in [107].

We cannot conclude without mentioning the “LHC connection”. After all, fundamental scalar fields, hitherto unseen, are assumedly involved in inflation, dark energy and other cosmological scenarios. It is widely believed that the Higgs particle will be observed after the few first stages of the LHC’s proper operation.

Some skepticism is also warranted on that. The reason is that “minimality” of the scalar sector of the Standard Model of particle physics is just a theoretical prejudice. This has been particularly emphasized by Strassler [108]. (Yes, *entes non sunt multiplicanda praeter necessitatem*. But Nature does not care for Ockham’s razor: who ordered the muon?)

There is the distinct possibility that something was overlooked at LEP and that the Higgs sector be considerably more complicated than in standard lore. Tension has been growing for a while between precision results and direct Higgs searches. The basic trouble was laid down by Chanowitz a few years ago [109]: if one eliminates from the precision electroweak data the (outlier) value of the forward-backward asymmetry into  $b$ -quarks, then the expected value for the Higgs mass drops to less than 50 GeV or so; with the mentioned outlier attributable to new physics. Otherwise, the overall fit is poor. This leads us to take *cum grano salis* the exclusion results at LEP. For instance, mixing with “hidden world” scalars leads to reduction to the standard Higgs couplings (consult [110] and references therein), in particular the  $ZZh$  coupling; and this could not be, and was not, ruled out by LEP for those relatively low energies. Other Higgs sector scenarios shielding the Higgs particle from detection have been discussed in [111, 112].

Recent experiment has made the situation even murkier: on Halloween night of 2008, ghostly (albeit rather abundant) multi-muon events at Fermilab were reported by (a majority segment of) the CDF collaboration [113]. A possible explanation for them invokes “new” light Higgs-like particles coupling relatively strongly to the “old” ones, and much less so to the SM fermions and MVB [114, 115]. There is also the possibility that the visible Higgs boson be rather *heavier* than expected, the discrepancy with the precision results being (somewhat brazenly) attributed to new physics [116, 117]. Then the *inert* Higgs boson would be a prime candidate for dark matter.

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